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Combinatorial constructions of modules for infinite-dimensional Lie algebras, II. Parafermionic space.

Galin Georgiev

Abstract

The standard modules for an affine Lie algebra $\hat{\mathfrak{g}}$ have natural subquotients called parafermionic spaces – the underlying spaces for the so-called parafermionic conformal field theories associated with $\hat{\mathfrak{g}}$.

We study the case $\hat{\mathfrak{g}} = \widehat{sl}(n+1, \mathbb{C})$ for any positive integral level $k \geq 2$. Generalizing the $\mathcal{Z}$-algebra approach of Lepowsky, Wilson and Primc, we construct a combinatorial basis for the parafermionic spaces in terms of colored partitions. The parts of these partitions represent "Fourier coefficients" of generalized vertex operators (parafermionic currents) and can be interpreted as statistically interacting quasi-particles of color i, $1 \leq i \leq n$, and charge s, $1 \leq s \leq k-1$. From a combinatorial point of view, these bases are essentially identical with the bases for level $k-1$ principal subspaces given in [GeI]. In the particular case of the vacuum module, the character (string function) associated with our basis is the formula of Kuniba, Nakanishi and Suzuki [KNS] conjectured in a Bethe Ansatz layout.

New combinatorial characters are established for the whole standard vacuum $\hat{\mathfrak{g}}$-modules.

0 Introduction

0.1

We begin with a short outline of some results from [GeI] which will be needed here.

Let $\mathfrak{g} := sl(n+1, \mathbb{C})$ with a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ and simple roots α_i, $1 \leq i \leq n$, where the indices reflect the roots' location on the Dynkin diagram (the considerations below can be carried out for any simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ of type A-D-E). Denote by $x_{\pm\alpha_i}$, $1 \leq i \leq n$, the Chevalley generators of $\mathfrak{g}$. Let $Q := \sum_{i=1}^{n} \mathbb{Z}\alpha_i$, $P := \sum_{i=1}^{n} \mathbb{Z}\Lambda_i$ be the root and weight lattice respectively, where Λ_i, $1 \leq i \leq n$, are the fundamental weights of g. For some formal variable t, consider the untwisted affinizations $\hat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$, $\bar{\mathfrak{n}} := \mathfrak{n} \oplus \mathbb{C}[t, t^{-1}]$ with a scaling operator $D := -td/dt$. Let $\hat{\Lambda}_i$, $0 \leq i \leq n$, be the fundamental weights of $\hat{\mathfrak{g}}$, so that $\hat{\Lambda}_i = \hat{\Lambda}_0 + \Lambda_i$ when $1 \leq i \leq n$. For another formal variable z, denote by $X_{\pm\alpha_i}(z) := \sum_{m\in\mathbb{Z}} (x_{\pm\alpha_i} \otimes t^m) z^{-m-1}$ the vertex operators

(bosonic currents) of charge one, corresponding to simple roots and their negatives. Define also currents of higher charge $r \in \mathbb{Z}_+$: $X_{\pm r\alpha_i}(z) := X_{\pm\alpha_i}(z)^r$ (cf. [GeI]). The operator-valued ("Fourier) coefficients of $X_{r\alpha_i}(z)$ are called quasi-particles of color i and charge r.

For a given positive integral level k (the eigenvalue of c), let $L(k\hat{\Lambda}_0)$ be the vacuum standard $\hat{\mathfrak{g}}$-module, i.e., the highest weight integrable module with highest weight $k\hat{\Lambda}_0$ (and highest weight vector $v(k\hat{\Lambda}_0)$). For the sake of simplicity, we shall consider in this Introduction only vacuum highest weights, although the results are proved for a larger class of highest weights. Denoting by $U(\cdot)$ universal enveloping algebra, recall from [FS] that $W(k\hat{\Lambda}_0) := U(\bar{\mathfrak{n}}) \cdot v(k\hat{\Lambda}_0)$ is called principal subspace of the standard module. We constructed in [GeI] a basis for the principal subspace: It is built by quasi-particles of colors i, $1 \le i \le n$, and charges s, $1 \le s \le k$, acting on the highest weight vector $v(k\hat{\Lambda}_0)$. A straightforward counting of the basis resulted in the character formula announced by Feigin and Stoyanovsky [FS]:

$$\operatorname{Tr} q^D \Big|_{W(k\hat{\Lambda}_0)} = \sum_{\substack{p_1^{(1)},\ldots,p_1^{(k)}\ge 0 \\ \cdots\cdots\cdots \\ p_n^{(1)},\ldots,p_n^{(k)}\ge 0}} \frac{q^{\frac{1}{2}\sum_{l,m=1,\ldots,n}^{s,t=1,\ldots,k} A_{lm}B^{st}p_l^{(s)}p_m^{(t)}}}{\prod_{i=1}^n \prod_{s=1}^k (q)_{p_i^{(s)}}}, \tag{0.1}$$

where $(A_{lm})_{l,m=1}^n$ is the Cartan matrix of $\mathfrak{g}$, $B^{st} := \min\{s,t\}$, $1 \le s,t \le k$, and for $p \in \mathbb{Z}_+$, $(q)_p := (1-q)(1-q^2)\cdots(1-q^p)$, $(q)_0 := 1$ (for the character of a principal subspace with more general highest weight, see formula (5.27) [GeI]). This formula was interpreted in the Introduction of [GeI] as a character for the Fock space of nk different free bosonic quasi-particles (n different colors and k different charges) with an additional two-particle interaction $A_{lm}B^{st}$ between a quasi-particle of color l and charge s and another quasi-particle of color m and charge t. This interaction can be interpreted as a statistical interaction in the sense of Haldane [H].

As indicated in [GeI], one can generate in a similar fashion a basis for the whole standard module, employing in addition quasi-particles corresponding to the negative simple roots. We shall see in Proposition 0.1 at the end this Introduction that it is sufficient to add only quasi-particles of charge k corresponding to the negative simple roots (we might also refer to the latter as to *quasi-antiparticles* of charge $-k$).

0.2

One of our objectives in this paper is the generalization (to higher rank affine Lie algebras) of the vertex operator construction of parafermionic spaces, given by Lepowsky and Primc [LP] for $\hat{\mathfrak{g}} = \widehat{sl}(2,\mathbb{C})$. Their construction was the untwisted version of the ground-breaking works on $\mathcal{Z}$-algebras of Lepowsky and Wilson [LW].

Consider the so-called $\hat{\mathfrak{g}} \supset \hat{\mathfrak{h}}$ coset subspace $L(k\hat{\Lambda}_0)^{\hat{\mathfrak{h}}^+}$ of $L(k\hat{\Lambda}_0)$ consisiting of all the $\hat{\mathfrak{h}}^+ := \mathfrak{h} \otimes t\mathbb{C}[t]$-invariants, i.e., the vectors annihilated by $\hat{\mathfrak{h}}^+$. The parafermionic

space $L(k\hat{\Lambda}_0)^{\hat{\mathfrak{h}}+}_{kQ}$ is defined as the spase of kQ-coinvariants in $L(k\hat{\Lambda}_0)^{\hat{\mathfrak{h}}^+}$, i.e., $L(k\hat{\Lambda}_0)^{\hat{\mathfrak{h}}+}_{kQ} := L(k\hat{\Lambda}_0)^{\hat{\mathfrak{h}}^+}/(\rho(kQ)-1)\cdot L(k\hat{\Lambda}_0)^{\hat{\mathfrak{h}}^+}$, where ρ is a natural action of the abelian group $kQ \cong Q$. There is a natural projection $\pi_{L(k\hat{\Lambda}_0)^{\hat{\mathfrak{h}}+}}$ such that $L(k\hat{\Lambda}_0)^{\hat{\mathfrak{h}}^+} = \rho(kQ)\cdot\pi_{L(k\hat{\Lambda}_0)^{\hat{\mathfrak{h}}+}}\cdot W(k\hat{\Lambda}_0)$ (cf. (1.4) and (2.39) below) – this explains why structural results for the principal subspace are easily carried to the $\hat{\mathfrak{g}} \supset \hat{\mathfrak{h}}$ subspace (and henceforth to the parafermionic space itself). For example, the bosonic current $X_{r\alpha_i}(z)$ has simply to be replaced by $\pi_{L(k\hat{\Lambda}_0)^{\hat{\mathfrak{h}}+}} \cdot X_{r\alpha_i}(z)$ – the latter equals up to a nonzero constant and a power of z the familiar parafermionic current $\Psi_{r\alpha_i}(z)$ (called generalized vertex operator in the general setting of [DL]) which by analogy is said to have a color i and charge r. Roughly speaking, one could think of the parafermionic current Ψ as obtained from the bosonic current X by factoring a *free* bosonic field (cf. (2.20)). As expected, the quasi-particles of color i and charge r in the parafermionic setting will be the operator-valued ("Fourier") coefficients of the current $\Psi_{r\alpha_i}(z)$. Note by the way that the parafermionic counterpart of the simple relation $X_{r\alpha_i}(z) := X_{\alpha_i}(z)^r$ is more sophisticated and involves the familiar binomial correction terms:

$$\Psi_{r\alpha_i}(z) = \left(\prod_{\substack{l,p=1 \\ l>p}}^{r} (z_l - z_p)^{\frac{\langle\alpha_i,\alpha_i\rangle}{k}} \right) \Psi_{\alpha_i}(z_r)\cdots\Psi_{\alpha_i}(z_1) \Bigg|_{z_r=\cdots=z_1=z}, \tag{0.2}$$

where the binomial terms are to be expanded in nonnegative inegral powers of the second variable (before the expression is restricted on the hyperplane $z_r = \cdots = z_1 = z$). There are other novelties in the parafermionic picture: The most important one is probably the new constraint

$$\Psi_{k\alpha_i}(z) = \text{const}\,\rho(k\alpha_i),\ \text{const} \in \mathbb{C}^\times,\ 1 \le i \le n, \tag{0.3}$$

which implies that the maximal allowed quasi-particle charge is reduced from k in the principal subspace picture to $k-1$ in the parafermionic picture. Another subtlety is that the scaling operator D has to be shifted in the parafermionic setting by $-D^{\hat{\mathfrak{h}}}$, where $D^{\hat{\mathfrak{h}}}$ is the rescaled 0th mode of the Virasoro algebra associated with the vertex operator algebra $U(\hat{\mathfrak{h}})\cdot v(k\hat{\Lambda}_0)$, $\hat{\mathfrak{h}} := \mathfrak{h}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}c$ of $\mathfrak{h}$ (cf. (2.13)).

We build a quasi-particle basis for $L(k\hat{\Lambda}_0)^{\hat{\mathfrak{h}}+}_{kQ}$ in section 4 (the corresponding basis for $L(k\hat{\Lambda}_0)^{\hat{\mathfrak{h}}^+}$ is then obtained by multiplying with $\rho(k\alpha)$, $\alpha \in Q$). As expected, it is essentially the same as the [GeI] quasi-particle basis for the level $k-1$ principal subspace $W((k-1)\hat{\Lambda}_0)$. The only difference is that the two-particle interaction between a quasi-particle of color l and charge s and another quasi-particle of color m and charge t has a "parafermionic " shift $-A_{lm}\frac{st}{k}$ and is therefore given by

$$A_{lm}\min\{s,t\} - A_{lm}\frac{st}{k} = A_{lm}A^{(-1)}_{st},\ 1 \le s,t \le k-1, \tag{0.4}$$

where $(A_{lm})^n_{l,m=1}$ is again the Cartan matrix of $\mathfrak{g} = sl(n+1,\mathbb{C})$ and $(A^{(-1)}_{st})^{k-1}_{s,t=1}$ is the inverse of the Cartan matrix of $sl(k,\mathbb{C})$. In other words, the character formula associated

with our basis is the one conjectured by Kuniba, Nakanishi and Suzuki [KNS]:

$$\mathrm{Tr}\, q^{D-D^{\hat{\mathfrak{h}}}}\Bigg|_{L(k\hat{\Lambda}_0)^{\hat{\mathfrak{h}}^+}_{kQ}} = \sum_{\substack{p_1^{(1)},\ldots,p_1^{(k-1)}\geq 0\\ \cdots\cdots\cdots\\ p_n^{(1)},\ldots,p_n^{(k-1)}\geq 0}} \frac{q^{\frac{1}{2}\sum_{l,m=1,\ldots,n}^{s,t=1,\ldots,k-1} A_{lm}A^{(-1)}_{st}p_l^{(s)}p_m^{(t)}}}{\prod_{i=1}^n\prod_{s=1}^{k-1}(q)_{p_i^{(s)}}}. \tag{0.5}$$

Note that a dilogarithm proof of this formula has already been announced by Kirillov [Kir], so it might be appropriate to emphasize that our goal here is not simply to find another proof of this beautiful formula, but to reveal the underlying conceptual structure. This in-depth approach, although painful, will pay off as we shall see for example in the subsequent Proposition 0.1. Note also that our arguments work for more general dominant highest weight (cf. (4.1) below)– the associated character formula is given in (5.7) Section 5. After an appropriate normalization, one immediately obtaines from the above formulas combinatorial expressions for the corresponding string functions.

Note that if we restrict our attention to the particular case $\mathfrak{g} = sl(2,\mathbb{C})$, the above formula (0.5) is the celebrated Lepowsky-Primc character [LP]. The underlying basis in [LP] is nevertheless very different from ours: Their construction employs only quasi-particles of charge one, governed by the so-called "difference 2 at distance $k-1$" condition, while we work with quasi-particles of charge r, $1 \leq r \leq k-1$, governed by a "difference $2r$ at distance 1" condition. A straightforward generalization of the Lepowsky-Primc parafermionic basis for $\mathfrak{g} = sl(3,\mathbb{C})$ was given in [P].

0.3

We continue with the higher level generalization of the new character formula for the level one standard module $L(\hat{\Lambda}_0)$ given in Proposition 0.1 [GeI] (for simplicity, we restrict ourselves to vacuum modules only).

As explained already, from the [GeI] basis for the principal subspace $W(k\hat{\Lambda}_0)$ we easily obtain a basis for the $\hat{\mathfrak{g}} \supset \hat{\mathfrak{h}}$ subspace $L(k\hat{\Lambda}_0)^{\hat{\mathfrak{h}}^+}$. The latter immediately implies a basis for the whole standard module $L(k\hat{\Lambda}_0) \cong U(\hat{\mathfrak{h}}) \cdot v(k\hat{\Lambda}_0) \otimes L(k\hat{\Lambda}_0)^{\hat{\mathfrak{h}}^+}$: The associated character formula is (cf. (0.1)):

$$\mathrm{Tr}\, q^{D}\Bigg|_{L(k\hat{\Lambda}_0)} = \frac{1}{(q)^n_\infty} \sum_{\substack{p_1^{(1)},\ldots,p_1^{(k-1)}\geq 0\\ \cdots\cdots\cdots\\ p_n^{(1)},\ldots,p_n^{(k-1)}\geq 0}} \frac{q^{\frac{1}{2}\sum_{l,m=1,\ldots,n}^{s,t=1,\ldots,k-1} A_{lm}B^{st}p_l^{(s)}p_m^{(t)}}}{\prod_{i=1}^n\prod_{s=1}^{k-1}(q)_{p_i^{(s)}}} \cdot \sum_{\alpha\in Q} q^{\frac{k}{2}\langle\alpha,\alpha\rangle + \langle\alpha, \sum_{i=1}^n r_i\alpha_i\rangle}, \tag{0.6}$$

where $r_i := \sum_{s=1}^{k-1} s p_i^{(s)}$ (cf. (5.15) for a more general highest weight). Note that the sum over Q incorporates the contribution of the factors $\rho(k\alpha)$, $\alpha \in Q$, and is easily expressed

in terms of the classical theta function of degree k:

$$\sum_{\alpha\in Q} q^{\frac{k}{2}\langle\alpha,\alpha\rangle+\langle\alpha,\mu\rangle} = q^{-\frac{\langle\mu,\mu\rangle}{2k}}\Theta_\mu(q) = q^{-\frac{\langle\mu,\mu\rangle}{2k}} \sum_{\gamma\in Q+\frac{\mu}{k}} q^{\frac{k}{2}\langle\gamma,\gamma\rangle}. \tag{0.7}$$

Although the presence of the theta function is somewhat intimidating, the above expression can still be interpreted as a trace along a combinatorial basis: This would be a semiinfinite monomial basis built up from quasi-particles corresponding to (positive) simple roots; cf. [FS] where the case $\mathfrak{g} = sl(2,\mathbb{C})$ is discussed.

But there is yet another – and probably the most natural – way to generate a basis for the whole standard module starting from the principal subspace: Simply add quasi-particles $x_{-k\alpha_i}(m)$ corresponding to negative simple roots (we shall call those anti-quasiparticles of charge $-k$) and take into account the identity

$$(z_2-z_1)^{r\langle\alpha_i,\alpha_i\rangle} X_{-k\alpha_i}(z_2)X_{r\alpha_i}(z_1)\Bigg|_{z_1=z_2=z} = \text{const}\, X_{-(k-r)\alpha_i}(z), \quad \text{const}\in\mathbb{C}^\times, \tag{0.8}$$

for every simple root α_i, $1\le i\le n$, and charge r, $1\le r\le k$. In particular, this identity implies that all the quasi-antiparticles of charge $-r$, $1\le r<k$, can be generated by anti-quasiparticles of charge $-k$ and usual quasi-particles. Moreover, when $r=k$, one gets an important new constraint between quasiparticles of charge k and anti-quasiparticles of charge $-k$ (the right-hand side is just a constant). Following the layout of [GeI], it is now not difficult to generate a basis for the whole standard module and write down the character formula associated with it:

Proposition 0.1 *One has the following q-character for the vacuum standard module at level k:*

$$\operatorname{Tr} q^D\Bigg|_{L(k\hat\Lambda_0)} = \sum_{\substack{p^{(1)}_{\pm1},\dots,p^{(k)}_{\pm1}\ge0\\ \dots\dots\dots\\ p^{(1)}_{\pm n},\dots,p^{(k)}_{\pm n}\ge0\\ p^{(s)}_{-i}=0\ \forall s<k,\,\forall i}} \frac{q^{\frac12\sum_{l,m=1,\dots,n}^{s,t=1,\dots,k} A_{lm}B^{st}(p^{(s)}_{+l}-p^{(s)}_{-l})(p^{(t)}_{+m}-p^{(t)}_{-m})+\sum_{l=1}^n p^{(k)}_{+l}p^{(k)}_{-l}}}{\prod_{i=1}^n\prod_{s=1}^k (q)_{p^{(s)}_{+i}}(q)_{p^{(s)}_{-i}}}, \tag{0.9}$$

where $(A_{lm})_{l,m=1}^n$ is the Cartan matrix of $\mathfrak{g}$, $B^{st} := min\{s,t\}$, $1\le s,t\le k$, and for $p\in\mathbb{Z}_+$, $(q)_p := (1-q)(1-q^2)\cdots(1-q^p)$, $(q)_0 := 1$.

In complete analogy with the level one particular case (cf. [GeI] Proposition 0.1), the above formula follows from (0.6) and the "Durfee rectangle" combinatorial identity [A]

$$\frac{1}{(q)_\infty} := \prod_{l\ge0}(1-q^l)^{-1} = \sum_{\substack{a,b\ge0\\ a-b=\text{const}}} \frac{q^{ab}}{(q)_a(q)_b}. \tag{0.10}$$

The basis underlying the above expression will be discussed in details somewhere else.

In the simplest particular case $n = k = 1$, the right-hand side of the above character reduces to

$$\sum_{p_{+1},p_{-1}\geq 0} \frac{q^{p_{+1}^2+p_{-1}^2-p_{+1}p_{-1}}}{(q)_{p_{+1}}(q)_{p_{-1}}},$$

a formula which appeared first in [FS].

0.4

Since the above Proposition 0.1 is the higher level generalization of Proposition 0.1 [GeI], the natural question arises, what is the higher level generalization of Proposition 0.2 [GeI], i.e., how to build a basis from intertwining operators corresponding to fundamental weights and their negatives (this question was posed long time ago by J. Lepowsky)? This remains a difficult open problem although a promising breakthrough in the simplest particular case $\hat{\mathfrak{g}} = \widehat{sl}(2,\mathbb{C})$ was recently made by Bouwknegt, Ludwig and Schoutens [BLS1] (cf. also [BPS], [I]): Using explicitly the easy to describe fusion algebra of the fusion category for $\widehat{sl}(2,\mathbb{C})$-modules, they built a basis of the above type (the so-called spinon basis) and derived the corresponding combinatorial character formula. As expected, the [BLS1] character is much more complicated than the particular $\widehat{sl}(2,\mathbb{C})$-case of formula (0.9). The reason is that (0.9) reflects a basis built up from usual (as opposed to intertwining!) vertex operators, namely, the ones corresponding to simple roots. Of course, the latter do not interchange different modules and for that matter, no knowledge of the fusion algebra is needed.

Let us note that the [BLS1] basis in principal gradation was identified with the basis proposed in [FIJKMY] (cf. [BLS2]). Moreover, the [BLS1] character formula was later independently derived using crystaline spinon basis, i.e., spinon basis for standard modules over $U_q(\widehat{sl}(2,\mathbb{C}))$ at $q = 0$ (cf. [NY], [ANOT]). No attempts have been made yet to crystalize the (q-deformation of the) basis underlying the above character formula (0.9).

Since the character (0.5) was originally conjectured by Kuniba, Nakanishi and Suzuki [KNS] from Thermodynamic Bethe Ansatz (TBA) considerations, it remains an open problem to understand the connection between our vertex operator construction and TBA. One possible approach, adopted by Melzer [M], Foda and Warnaar [FW], [W] in the context of Virasoro algebra modules, is through "finitization" of the above q-series, i.e., representing them as a limit of certain q-polynomials.

Finally, from the point of view of our approach, one of the most important and challenging unsolved problems is of course to find a generalization for arbitrary (nonintegral) levels and highest weights and also, to build vertex operator bases for other coset spaces. This will in particular provide vertex operator constructions for many other modules over W-algebras. It will also cast some light upon the connection with the approach of Berkovich and McCoy [BM] to Virasoro algebra modules from the minimal series and to branching functions associated with various other coset spaces considered in [KM], [KKMM], [DKKMM], [KMM], [WP], [BG].

0.5

In Section 1 we define the $\hat{\mathfrak{g}} \supset \hat{\mathfrak{h}}$ coset subspace of a standard $\hat{\mathfrak{g}}$-module ($g = sl(n+1, \mathbb{C})$) at any positive integral level $k \geq 2$. The parafermionic space is defined as its natural quotientspace. Section 2 introduces relative vertex operators (parafermionic currents). Section 3 explains how the notions of quasi-particle and quasi-particle momomial (from part I) have to be modified in the current setting. In Section 4 we build a quasi-particle monomial basis for the parafermionic space. Section 5 presents the corresponding character formulas for the parafermionic space (string function) and for the whole standard module. Tables 1 and 2 in the Appendix illustrate Examples 4.1 and 4.2, Section 4.

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1 The $\hat{\mathfrak{g}} \supset \hat{\mathfrak{h}}$ coset subspace and its parafermionic quotientspace

Although the exposition below is selfcontained, we shall follow the framework of [GeI] and use many of its notations, definitions and results. We continue working with $\hat{\mathfrak{g}} = \widehat{sl}(n+1, \mathbb{C})$ and a level k dominant integral highest weight $\hat{\Lambda} = k_0\hat{\Lambda}_0 + k_j\hat{\Lambda}_j$ for some j, $1 \leq j \leq n$, and $k_0, k_j \in \mathbb{N}$, $k_0 + k_j = k \geq 2$, where $\{\hat{\Lambda}_l\}_{l=0}^n$ are the fundamental weights of $\hat{\mathfrak{g}}$. In other words, $\hat{\Lambda} = k\hat{\Lambda}_0 + \Lambda$, where $\Lambda = k_j\Lambda_j \in P$ and $\{\Lambda_l\}_{l=1}^n$ are the fundamental weights of $\mathfrak{g}$. (All the objects introduced in Sections 1,2 and 3 have obvious generalizations for any dominant integral highest weight.)

Recall that $\hat{\mathfrak{h}} := \mathfrak{h} \otimes \mathbb{C}[t^{-1}, t] \oplus \mathbb{C}c \subset \hat{\mathfrak{g}}$ is the affinization of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The $\hat{\mathfrak{g}} \supset \hat{\mathfrak{h}}$ *coset subspace* $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}$ of the standard $\hat{\mathfrak{g}}$-module $L(\hat{\Lambda})$ is defined as the vacuum subspace for $\hat{\mathfrak{h}}^+$, i.e., the linear span of all the vectors $\{v \in L(\hat{\Lambda}) | \hat{\mathfrak{h}}^+ \cdot v = 0\}$, where $\hat{\mathfrak{h}}^+ := \mathfrak{h} \otimes t\mathbb{C}[t]$. A concrete realization of this space can be given in terms of the vertex operator construction of basic modules: Set $M(k) := U(\hat{\mathfrak{h}}) \otimes_{U(\mathfrak{h}\otimes\mathbb{C}[t]\oplus\mathbb{C}c)} \mathbb{C}$, with $\mathfrak{h} \otimes \mathbb{C}[t]$ acting trivially on $\mathbb{C}$ and c acting as k (cf. [GeI] Preliminaries). Let $V_P := M(1) \otimes \mathbb{C}[P]$ and recall that by the classical interpretation of V_P as a $\hat{\mathfrak{g}}$-module [FK], [S], one has $V_P \cong \oplus_{j=0}^n L(\hat{\Lambda}_j)$ (cf. [GeI], Section 2). Therefore, the standard module $L(\hat{\Lambda})$ can be explicitly generated by the universal eneveloping algebra $U(\hat{\mathfrak{g}})$ acting on the highest weight vector $v(\hat{\Lambda}) \subset V_P^{\otimes k}$ through the $(k-1)$-iterate Δ^{k-1} of the standard coproduct Δ (cf. [GeI], Sections 2, 5; no confusion can arise from the fact that we denote by the same letter Δ the root system of $\mathfrak{g}$). Then the $\hat{\mathfrak{g}} \supset \hat{\mathfrak{h}}$ coset subspace of $L(\hat{\Lambda})$ is of course

$$L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+} := \mathrm{span}_{\mathbb{C}}\{v \in L(\hat{\Lambda}) = U(\hat{\mathfrak{g}}) \cdot v(\hat{\Lambda}) | \hat{\mathfrak{h}}^+ \cdot v = 0\} \subset V_P^{\otimes k}. \tag{1.1}$$

Note how easy it is to reconstruct the whole space $L(\hat{\Lambda})$ from $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}$ because of the

canonical isomorphisms of D-graded linear spaces [LW]

$$\begin{array}{rcl} U(\hat{\mathfrak{h}}^-) \otimes L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+} & \xrightarrow{\sim} & L(\hat{\Lambda}) \\ h \otimes u & \mapsto & h \cdot u, \end{array} \tag{1.2}$$

and

$$S(\hat{\mathfrak{h}}^-) \cong U(\hat{\mathfrak{h}}^-) \cong M(k), \tag{1.3}$$

where $\hat{\mathfrak{h}}^- := \hat{\mathfrak{h}} \otimes t^{-1}\mathbb{C}[t^{-1}]$, $S(\cdot)$ is a symmetric algebra and $M(k)$ was defined above. One can therefore consider the projection

$$\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} : L(\hat{\Lambda}) \to L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}, \tag{1.4}$$

given by the corresponding direct sum decomposition

$$L(\hat{\Lambda}) = L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+} \oplus \hat{\mathfrak{h}}^- U(\hat{\mathfrak{h}}^-) \cdot L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}. \tag{1.5}$$

We can further reduce the $\hat{\mathfrak{g}} \supset \hat{\mathfrak{h}}$ coset subspace using its natural structure of module for the root lattice Q: Observe that the map $\alpha \mapsto e^\alpha$, $\alpha \in Q$, defines an action of Q on $V_P = M(1) \otimes \mathbb{C}[P]$ (cf. [GeI], Section 2) by restriction to the right factor. It thus commutes with the action of $\hat{\mathfrak{h}}^+$ on V_P, which affects only the left factor. Define a diagonal action of the sublattice $kQ \subset Q$ ($kQ \cong Q$) on $V_P^{\otimes k}$:

$$k\alpha \mapsto \rho(k\alpha) := \underbrace{e^\alpha \otimes \ldots \otimes e^\alpha}_{k\ factors},\ \alpha \in Q. \tag{1.6}$$

Note that it commutes with the action $\Delta^{k-1}(\hat{\mathfrak{h}}^+)$ of $\hat{\mathfrak{h}}^+$ and hence preserves $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+} \subset V_P^{\otimes k}$. Following [DL], Ch. 4, we consider the space of kQ-coinvariants in the kQ-module $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}$:

$$L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ} := L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+} \Big/ \mathrm{Span}\{(\rho(k\alpha) - 1) \cdot v | \alpha \in Q,\ v \in L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}\} \ . \tag{1.7}$$

This quotientspace is called *parafermionic space* (of highest weight $\hat{\Lambda}$) because it is a building block for the so-called parafermionic conformal field theories [ZF], [G], [LP], [DL]. We shall denote by

$$\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}} \ : L(\hat{\Lambda}) \to L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ} \tag{1.8}$$

the composition of $\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}}$ from (1.4) and the obvious projection of $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}$ onto $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}$.

Our goal here is to construct a quasi-particle basis for the parafermionic space $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}$, modifying appropriately the quasi-particle basis for the principal subspace $W(\hat{\Lambda}) \subset L(\hat{\Lambda})$ built in [GeI]. Observe that if we denote by $L_\mu(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}$ the weight subspace of $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}$, corresponding to $\mu \in P$ and by $L_\mu(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}$ its isomorphic (as graded linear space) image in the quotient $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}$, we have

$$\rho(k\alpha) \cdot L_\mu(\hat{\Lambda})^{\hat{\mathfrak{h}}^+} = L_{\mu+k\alpha}(\hat{\Lambda})^{\hat{\mathfrak{h}}^+},\ \alpha \in Q,\ \mu \in P, \tag{1.9}$$

and

$$L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ} = \coprod_{\mu\in\Lambda+Q/kQ} L_\mu(\Lambda)^{\hat{\mathfrak{h}}^+}_{kQ} \cong \coprod_{\mu\in\Lambda+Q/kQ} L_\mu(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}. \tag{1.10}$$

Therefore a basis for $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}$ furnishes automatically a basis for the whole $\hat{\mathfrak{g}} \supset \hat{\mathfrak{h}}$ coset subspace $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+} = \coprod_{\mu\in\Lambda+Q} L_\mu(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}$. (No need to say, the tensor product decomposition (1.2) then provides a basis for the whole standard module $L(\hat{\Lambda})$.)

The trivial action ρ of kQ on the parafermionic space is very suggestive for considering a natural action (denoted by the same letter ρ) of the finite abelian group Q/kQ on $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}$ (cf. [DL], Ch. 6): Set

$$\rho(\alpha)\cdot v := e^{2\pi i \frac{\langle\alpha,\mu\rangle}{k}} v,\ \alpha\in Q/kQ,\ v\in L_\mu(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}. \tag{1.11}$$

The characters $e^{2\pi i\frac{\langle\cdot,\mu\rangle}{k}}$, $\mu\in\Lambda+Q/kQ$, are indeed the simple characters of the group Q/kQ, in other words, the decomposition (1.10) coincides with the character-space decomposition of $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}$ under the above action of Q/kQ.

2 Generalized vertex operators (parafermionic currents)

In this Section, we shall largely use the methods of generalized vertex operator algebra theory developed by Dong and Lepowsky [DL] (cf. also [FLM]).

Recall that the main protagonist in the level k setting of [GeI] was the vertex operator (bosonic current)

$$X_\beta(z) := \Delta^{k-1}(Y(e^\beta,z)) = \tag{2.1}$$
$$= \underbrace{Y(e^\beta,z)\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}}_{k\,factors} + \underbrace{\mathbf{1}\otimes Y(e^\beta,z)\otimes\cdots\otimes\mathbf{1}}_{k\,factors} + \cdots + \underbrace{\mathbf{1}\otimes\cdots\otimes\mathbf{1}\otimes Y(e^\beta,z)}_{k\,factors},$$

where $\beta\in\Delta$,

$$Y(e^\beta,z) := E^-(-h_\beta,z)E^+(-h_\beta,z)\otimes e^\beta z^{h_\beta}\varepsilon_\beta, \tag{2.2}$$

$$E^\pm(h,z) := \exp\left(\sum_{m\geq 1} h(\pm m)\frac{z^{\mp m}}{\pm m}\right),\ h\in\mathfrak{h}, \tag{2.3}$$

and $\varepsilon_\beta := \varepsilon(\beta,\cdot)$, $\varepsilon : P\times P\to\mathbb{C}^\times$ being a 2-cocycle on the weight lattice P (cf. [GeI] Sections 1 - 3). The "Fourier coefficients" of this vertex operator are given by its expansion

$$X_\beta(z) =: \sum_{m\in\mathbb{Z}} x_\beta(m) z^{-m-1}, \tag{2.4}$$

on powers of the formal variable z. The action of the affine algebra $\hat{\mathfrak{g}}$ on the standard module $L(\hat{\Lambda}) = U(\hat{\mathfrak{g}})\cdot v(\hat{\Lambda})$ is then given by $x_\beta\otimes t^m := x_\beta(m)$, $\beta\in\Delta,\ m\in\mathbb{Z}$.

The Jacobi identity for the vertex operator algebra $L(k\hat{\Lambda}_0)$ generated by these currents, implies the usual formulas for the currents of higher charge $r \in \mathbb{Z}_+$:

$$X_{r\beta}(z) := \underbrace{X_\beta(z)\cdots X_\beta(z)}_{r factors} := Y(x_\beta(-1)^r \cdot v(k\hat{\Lambda}_0), z), \tag{2.5}$$

where $v(k\hat{\Lambda}_0)$ is the vacuum highest weight vector at level k (cf. for example [DL] Proposition 13.16). Note that since the product $X_\beta(z_2)X_\beta(z_1)$ is not singular on the hyperplane $z_2 = z_1 = z$ (cf. [GeI] (2.8)), one does not need to "regularize" it with powers of $z_2 - z_1$ before one sets $z_2 = z_1 = z$. This is not the case anymore in the parafermionic picture – see (2.19) below. We actually showed in [GeI] that the currents corresponding to the simple roots $\alpha_i,\ 1 \le i \le n$, are enough to build a basis of the principal subspace $W(\hat{\Lambda})$: their "Fourier coefficients" played the role of quasi-particles and the basis was generated by quasi-particle monomials (from an appropriate completion of the ordered product $U := U(\bar{\mathfrak{n}}_{\alpha_n})\cdots U(\bar{\mathfrak{n}}_{\alpha_1})$ acting on the highest weight vector $v(\hat{\Lambda})$).

Switching now our attention from the principal subspace $W(\hat{\Lambda})$ to the parafermionic space $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}$, we immediately observe that the above operators do not even preserve the vacuum subspace $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}$, let alone its quotientspace $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}$! Fortunately, one can perform a small cosmetic operation and fix this problem (cf. [LP], [ZF], [G], [DL]): For every $\beta \in \Delta$, replace $X_\beta(z)$ by the *relative vertex operator* (called also *parafermionic current*) on $V_P^{\otimes k}$

$$\Psi_\beta(z) := \left(\underbrace{E^-(\frac{h_\beta}{k}, z) \otimes \cdots \otimes E^-(\frac{h_\beta}{k}, z)}_{k\ factors}\right) X_\beta(z) \tag{2.6}$$

$$\left(\underbrace{z^{-\frac{h_\beta}{k}}\varepsilon_\beta^{-\frac{1}{k}} \otimes \cdots \otimes z^{-\frac{h_\beta}{k}}\varepsilon_\beta^{-\frac{1}{k}}}_{k\ factors}\right)\left(\underbrace{E^+(\frac{h_\beta}{k}, z) \otimes \cdots \otimes E^+(\frac{h_\beta}{k}, z)}_{k\ factors}\right).$$

Note that the coefficients of this operator lie in an appropriate completion of $U(\hat{\mathfrak{g}})$, i.e., they are infinite sums which are truncated when acting on modules from the category $\mathcal{O}$. We do not have to specify here exactly which root of unity is to be taken in $\varepsilon_\beta^{-\frac{1}{k}}$ as long as it is the same on all tensor slots. The first and the last factor on right-hand side of the above definition, together with the k^{th} root of formula (2.7) [GeI], ensure that

$$\left[\hat{\mathfrak{h}}^+, \Psi_\beta(z)\right] = \left[\hat{\mathfrak{h}}^-, \Psi_\beta(z)\right] = 0, \tag{2.7}$$

i.e., the relative vertex operator indeed preserves the vacuum space $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}$ (recall that $\hat{\mathfrak{h}}$ acts on $V_P^{\otimes k}$ through the iterated coproduct Δ^{k-1}). But it is the term in the middle (not present in the $\mathcal{Z}$-operators of Lepowsky and Primc [LP])

$$\underbrace{z^{-\frac{h_\beta}{k}}\varepsilon_\beta^{-\frac{1}{k}} \otimes \cdots \otimes z^{-\frac{h_\beta}{k}}\varepsilon_\beta^{-\frac{1}{k}}}_{k\ factors}, \tag{2.8}$$

which guarantees that

$$[\rho(k\alpha), \Psi_\beta(z)] = 0,\ \alpha, \beta \in Q, \tag{2.9}$$

and therefore Ψ is well defined on the parafermionic space $L(\hat{\Lambda})^{\hat{\mathfrak{h}}+}_{kQ}$. It will be clear from the context whether Ψ acts on $L(\hat{\Lambda})^{\hat{\mathfrak{h}}+}$ or $L(\hat{\Lambda})^{\hat{\mathfrak{h}}+}_{kQ}$.

Remark 2.1 We should note that the nonzero numerical coset correction $\varepsilon_\beta^{-\frac{1}{k}} \otimes \cdots \otimes \varepsilon_\beta^{-\frac{1}{k}}$ is not present in the definition adopted in [DL]. As a result, the relative vertex operator in [DL] does not commute with $\rho(kQ)$ but only with $\rho(2kQ)$ and one is forced to consider a larger parafermionic space associated with the finite group $Q/2kQ$.

The components ("Fourier coefficients") of Ψ will be indexed in the very same fashion as the coefficients of X, but due to the term (2.8), their indices will typically be rational numbers rather than integers: For every $\beta \in \Delta$, set

$$\Psi_\beta(z)\Bigg|_{L_\mu(\hat{\Lambda})^{\hat{\mathfrak{h}}+}} =: \sum_{m\in\mathbb{Z}+\frac{\langle\beta,\mu\rangle}{k}} \psi_\beta(m) z^{-m-1}\Bigg|_{L_\mu(\hat{\Lambda})^{\hat{\mathfrak{h}}+}}, \tag{2.10}$$

where $L_\mu(\hat{\Lambda})^{\hat{\mathfrak{h}}+}$ is the μ-weight subspace of $L(\hat{\Lambda})^{\hat{\mathfrak{h}}+}$, $\mu \in P$ (cf. for example [DL] (6.52); note that the operator $\psi_\beta(m)$ is defined only on those μ-weight subspaces for which $m \in \mathbb{Z} + \frac{\langle\beta,\mu\rangle}{k}$). This definition is designed so that the coefficients $\psi_\beta(m)$ can also be thought of as operators on the parafermionic space $L(\hat{\Lambda})^{\hat{\mathfrak{h}}+}_{kQ}$:

$$[\rho(k\alpha), \psi_\beta(m)] = 0,\ \alpha, \beta \in Q. \tag{2.11}$$

We call $-m - \frac{1}{k}$ (rather than $-m$) a *conformal energy* of $\psi_\beta(m)$. This is because the Virasoro algebra generators are shifted on a coset space: In our parafermionic setting, we have to replace the grading operator D of the vertex operator algebra $L(k\hat{\Lambda}_0)$ (cf. [GeI], Preliminaries) by $D - D^{\hat{\mathfrak{h}}}$, where

$$\left[D - D^{\hat{\mathfrak{h}}}, \psi_\beta(m)\right] = -(m + \frac{1}{k})\psi_\beta(m), \tag{2.12}$$

cf. e.g. [DL], (6.42) and (14.87). (Formula (2.12) is the parafermionic counterpart of the commutation relation $[D, x_\beta(m)] = -mx_\beta(m)$.) In other words,

$$D^{\hat{\mathfrak{h}}}\Bigg|_{L_\mu(\hat{\Lambda})^{\hat{\mathfrak{h}}+}} = L_0^{\hat{\mathfrak{h}}} - \frac{\langle\Lambda,\Lambda\rangle}{2k}\Bigg|_{L_\mu(\hat{\Lambda})^{\hat{\mathfrak{h}}+}} = \frac{\langle\mu,\mu\rangle}{2k} - \frac{\langle\Lambda,\Lambda\rangle}{2k}\Bigg|_{L_\mu(\hat{\Lambda})^{\hat{\mathfrak{h}}+}}, \tag{2.13}$$

where $L_0^{\hat{\mathfrak{h}}}$ is the 0th mode of the Virasoro algebra associated with the vertex operator algebra $U(\hat{\mathfrak{h}}) \cdot v(k\hat{\Lambda}_0) \cong M(k)$ (cf. [DL] (14.52) and [K] Remark 12.8 for example).

Pivotal for our arguments will be the observation that the "Fourier coefficients" of $\Psi_\beta(m)$ can be tied with the "Fourier coefficients" of $X_\beta(z)$ in a very simple way: For any given weight $\mu \in P$, one has from the very definitions (2.4), (2.6) and (2.10)

$$\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot x_\beta(m) \Big|_{L_\mu(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} = \mathrm{const}\, \psi_\beta(m + \frac{\langle \beta, \mu \rangle}{k}) \Big|_{L_\mu(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} , \tag{2.14}$$

where $m \in \mathbb{Z}$, const $\in \mathbb{C}^\times$ and the natural projection $\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} : L(\hat{\Lambda}) \to L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}$ was introduced in (1.4), (1.5). Note that this identity does not make much sense if $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}$ is replaced by $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}$ and if ψ is thought of as an operator on $L_\mu(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}$ because $\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}} \cdot x_\beta(m)$ does not really act on the quotientspace $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}$. This is one of the reasons why we shall often use $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}$ as a mediator between the principal subspace and the parafermionic space.

Observe that one can reverse the definition (2.6) and thus "isolate" the part of the vertex operator $X_\beta(z)$ which acts on the $\hat{\mathfrak{g}} \supset \hat{\mathfrak{h}}$ subspace $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+} \subset L(\hat{\Lambda})$:

$$X_\beta(z) = \left(\underbrace{E^-(-\frac{h_\beta}{k}, z) \otimes \cdots \otimes E^-(-\frac{h_\beta}{k}, z)}_{k \ factors} \right) \Psi_\beta(z) \tag{2.15}$$

$$\left(\underbrace{z^{\frac{h_\beta}{k}} \varepsilon_\beta^{\frac{1}{k}} \otimes \cdots \otimes z^{\frac{h_\beta}{k}} \varepsilon_\beta^{\frac{1}{k}}}_{k \ factors} \right) \left(\underbrace{E^+(-\frac{h_\beta}{k}, z) \otimes \cdots \otimes E^+(-\frac{h_\beta}{k}, z)}_{k \ factors} \right).$$

Lemma 2.1 *The $\hat{\mathfrak{g}} \supset \hat{\mathfrak{h}}$ subspace $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}$ (and therefore the parafermionic space $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}$) is generated by the operators $\{\psi_\beta(m) | \beta \in \Delta\}$ acting on the highest weight vector $v(\hat{\Lambda})$, i.e.,*

$$L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+} = \mathrm{Span}_{\mathbb{C}} \left\{ \psi_{\beta_r}(m_r) \cdots \psi_{\beta_1}(m_1) \cdot v(\hat{\Lambda}) | \beta_l \in \Delta,\ 1 \leq l \leq r \right\}. \tag{2.16}$$

Proof Assume the opposite and using (2.15) and (1.2), arrive at a contradiction with the irreducibility of $L(\hat{\Lambda})$ (cf. [DL] Proposition 14.9). □

Note by the way that for $\beta \in \Delta$,

$$\rho(k\beta) \cdot v(\hat{\Lambda}) = \mathrm{const}\, x_\beta(-1 - \langle \beta, \Lambda \rangle) x_\beta(-1)^{k-1} \cdot v(\hat{\Lambda}) = \tag{2.17}$$

$$= \mathrm{const}'\, \pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot x_\beta(-1 - \langle \beta, \Lambda \rangle) x_\beta(-1)^{k-1} \cdot v(\hat{\Lambda}) =$$

$$= \text{const}'' \, \pi_{L(\hat\Lambda)^{\hat{\mathfrak h}+}} \cdot x_\beta(-1-\langle\beta,\Lambda\rangle) \left(\pi_{L(\hat\Lambda)^{\hat{\mathfrak h}+}} \cdot x_\beta(-1)\right)^{k-1} \cdot v(\hat\Lambda)$$

for some nonzero constants. Since $v(\hat\Lambda)$ is an eigenvector for $D - D^{\hat{\mathfrak h}}$ (cf. (2.13)), one can compute in a straightforward fashion from (2.12), (2.14) and (2.17) that $D - D^{\hat{\mathfrak h}}$ and $\rho(k\alpha)$, $\alpha \in Q$, commute when acting on a highest weight vector. It follows immediately from (2.11), (2.12) and Lemma 2.1 that $\rho(k\alpha)$ and $D - D^{\hat{\mathfrak h}}$ commute on $L(\hat\Lambda)^{\hat{\mathfrak h}^+}$. But $\rho(k\alpha)$ acts nontrivially only on the right factor of the decomposition (1.2), hence

$$[D - D^{\hat{\mathfrak h}}, \rho(k\alpha)] = 0, \ \alpha \in Q. \tag{2.18}$$

Let us continue now with the parafermionic currents of higher charge (well known in the physics literature – cf. e.g. [ZF], [G]): For a given $\beta \in \Delta$ and $r \in \mathbb{Z}_+$, we call a *parafermionic current of charge* r the generating function

$$\Psi_{r\beta}(z) := \left(\prod_{\substack{l,p=1 \\ l>p}}^{r} (z_l - z_p)^{\frac{\langle\beta,\beta\rangle}{k}} \right) \Psi_\beta(z_r) \cdots \Psi_\beta(z_1) \Bigg|_{\, z_r = \cdots = z_1 = z} , \tag{2.19}$$

where the binomial terms are to be expanded in nonnegative inegral powers of the second variable (before the expression is restricted on the hyperplane $z_r = \cdots = z_1 = z$). This generating function is well defined when acting on a highest weight module because the binomial terms cancel exactly the singularities related to the noncommutativity of the first and the last correction factors in the definition (2.6) of Ψ (cf. [GeI] (2.7), (2.8)). In other words, the above expression can be rewritten as

$$\Psi_{r\beta}(z) = \text{const} \left(\underbrace{E^-(\frac{h_\beta}{k}, z) \otimes \cdots \otimes E^-(\frac{h_\beta}{k}, z)}_{k\ factors} \right)^r X_{r\beta}(z) \tag{2.20}$$

$$\left(\underbrace{z^{-\frac{h_\beta}{k}} \varepsilon_\beta^{-\frac{1}{k}} \otimes \cdots \otimes z^{-\frac{h_\beta}{k}} \varepsilon_\beta^{-\frac{1}{k}}}_{k\ factors} \right)^r \left(\underbrace{E^+(\frac{h_\beta}{k}, z) \otimes \cdots \otimes E^+(\frac{h_\beta}{k}, z)}_{k\ factors} \right)^r ,$$

where const $\in \mathbb{C}^\times$ and $X_{r\beta}(z) = X_\beta(z)^r$ is the bosonic current of charge r from (2.5). It will be clear from the context whether Ψ acts on $L(\hat\Lambda)^{\hat{\mathfrak h}^+}$ or $L(\hat\Lambda)^{\hat{\mathfrak h}^+}_{kQ}$.

The generating function $\Psi_{r\beta}(z)$ is the parafermionic counterpart of $X_{r\beta}(z)$ in the following sense: Recall that $X_{r\beta}(z)$ is the vertex operator corresponding to the vector $x_\beta(-1)^r \cdot v(k\hat\Lambda_0)$ in the vertex operator algebra $L(k\hat\Lambda_0)$ (cf. (2.5)). According to (2.14), the projection of this vector on the parafermionic space is

$$\pi_{L(\hat\Lambda)^{\hat{\mathfrak h}+}} \cdot x_\beta(-1)^r \cdot v(k\hat\Lambda_0) = \text{const} \underbrace{\psi_\beta(-1 + \frac{\langle\beta,(r-1)\beta\rangle}{k}) \cdots \psi_\beta(-1)}_{r\, factors} \cdot v(k\hat\Lambda_0), \tag{2.21}$$

(the nonzero const equals one if we assume without losing generality that ε is bimultiplicative and $\varepsilon(\alpha,\alpha)=\varepsilon(\alpha,\alpha)^{\frac{1}{k}}=1$). But the Jacobi identity for the generalized (in the sense of [DL]) vertex operator algebra $L(k\hat{\Lambda}_0)^{\hat{\mathfrak{h}}+}_{kQ}$ easily implies that $\Psi_{r\beta}(z)$ is the vertex operator corresponding to this last vector, i.e.,

$$\Psi_{r\beta}(z) = Y(\underbrace{\psi_\beta(-1+\frac{\langle\beta,(r-1)\beta\rangle}{k})\cdots\psi_\beta(-1)}_{r\,factors}\cdot v(k\hat{\Lambda}_0), z) \tag{2.22}$$

(one can for example use repeatedly Proposition 14.29 [DL] which under the above assumptions for ε is still true even for our slightly modified Ψ, provided the notations are appropriately adjusted; be aware that the above $Y(\cdot,z)$ is not the same as $Y(\cdot,z)$ in (2.5) since these are vertex operators in two *different* vertex operator algebras).

In view of the correspondence between parafermionic and bosonic currents, the most natural generalization of the definition (2.10) of "Fourier coefficients" for higher-charge parafermionic currents is (cf. [GeI] (3.7))

$$\Psi_{r\beta}(z)\Bigg|_{L_\mu(\hat{\Lambda})^{\hat{\mathfrak{h}}+}} =: \sum_{m\in\mathbb{Z}+\frac{\langle r\beta,\mu\rangle}{k}} \psi_{r\beta}(m)z^{-m-r}\Bigg|_{L_\mu(\hat{\Lambda})^{\hat{\mathfrak{h}}+}}, \tag{2.23}$$

where $\beta\in\Delta,\ r\in\mathbb{Z}_+$ and $L_\mu(\hat{\Lambda})^{\hat{\mathfrak{h}}+}$ is as always the μ-weight subspace of $L(\hat{\Lambda})^{\hat{\mathfrak{h}}+}$, $\mu\in P$. Note that $\psi_{r\beta}(m)$ is a well defined operator on $L_\mu(\hat{\Lambda})^{\hat{\mathfrak{h}}+}_{kQ}$ with μ, such that $m\in\mathbb{Z}+\frac{\langle r\beta,\mu\rangle}{k}$, because $\psi_{r\beta}(m)$ commutes with the action ρ of kQ. Moreover, by (2.12) and the definitions (2.19), (2.23), the conformal energy of $\psi_{r\beta}(m)$ is

$$\begin{aligned}\left[D-D^{\hat{\mathfrak{h}}},\psi_{r\beta}(m)\right] &= -(m+\frac{1}{k}(r+\langle\beta,\beta\rangle\begin{pmatrix} r\\ 2\end{pmatrix})\psi_{r\beta}(m) = \\ &= -(m+\frac{r^2}{k})\psi_{r\beta}(m).\end{aligned} \tag{2.24}$$

Analogously to the charge-one situation (2.14), one can find for every operator $x_{r\beta}(m)$ its parafermionic counterpart (acting on a given weight subspace of $L(\hat{\Lambda})^{\hat{\mathfrak{h}}+}$) by composing it with the projection $\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}+}}$:

$$\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}+}}\cdot x_{r\beta}(m)\Bigg|_{L_\mu(\hat{\Lambda})^{\hat{\mathfrak{h}}+}} = \text{const}\,\psi_{r\beta}(m+\frac{\langle r\beta,\mu\rangle}{k})\Bigg|_{L_\mu(\hat{\Lambda})^{\hat{\mathfrak{h}}+}}, \tag{2.25}$$

where const $\in\mathbb{C}^\times$.

We are now in position to formulate a key relation which – simple and beautiful as it is – explains both the similarity and the difference between the structure of the parafermionic space $L(\hat{\Lambda})^{\hat{\mathfrak{h}}+}_{kQ}$ and the principal space $W(\hat{\Lambda})$.

Proposition 2.1 *For every $\beta \in \Delta$ and $r \in \mathbb{Z}_+,\ 1 \le r \le k$, one has*

$$\Psi_{r\beta}(z) = \mathrm{const}\,\rho(k\beta)\Psi_{-(k-r)\beta}(z), \tag{2.26}$$

const $\in \mathbb{C}^\times$, *(cf. (1.6)), i.e.,*

$$\Psi_{r\beta}(z) = \mathrm{const}\,\Psi_{-(k-r)\beta}(z), \tag{2.27}$$

const $\in \mathbb{C}^\times$, *as operators on* $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}$.

Proof Follows from a direct computation employing the commutation relation [GeI] (2.7) (in complete analogy with [LP] Theorem 5.6 for example). □

This in particular implies that products of coefficients of parafermionic currents associated with positive roots are enough for generating the parafermionic space (when acting on the vacuum vector; cf. Lemma 2.1 and (2.19)). In other words, in complete analogy with the principal (sub)space

$$W(\hat{\Lambda}) := \mathrm{Span}_{\mathbb{C}}\left\{x_{\beta_r}(m_r)\cdots x_{\beta_1}(m_1)\cdot v(\hat{\Lambda})|\beta_l \in \Delta_+,\ 1 \le l \le r\right\}, \tag{2.28}$$

one has

$$L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+} = \rho(kQ)\cdot \mathrm{Span}_{\mathbb{C}}\left\{\psi_{\beta_r}(m_r)\cdots \psi_{\beta_1}(m_1)\cdot v(\hat{\Lambda})|\beta_l \in \Delta_+,\ 1 \le l \le r\right\} \tag{2.29}$$

and hence

$$L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ} = \mathrm{Span}_{\mathbb{C}}\left\{\psi_{\beta_r}(m_r)\cdots \psi_{\beta_1}(m_1)\cdot v(\hat{\Lambda})|\beta_l \in \Delta_+,\ 1 \le l \le r\right\}. \tag{2.30}$$

Despite this similarity, there is an important difference between the parafermionic space and the principal (sub)space which is encoded in the particular case of Proposition 2.1 for $r = k$:

$$\Psi_{k\beta}(z) = \mathrm{const}\,\rho(k\beta), \tag{2.31}$$

const $\in \mathbb{C}^\times,\ \beta \in \Delta$, i.e.,

$$\Psi_{k\beta}(z) = \mathrm{const} \in \mathbb{C}^\times \tag{2.32}$$

when acting on the parafermionic space $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}$. In other words, one component of $\Psi_{k\beta}(z)$ acts as a nonzero constant on the parafermionic space $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}$ and all the other components vanish on it. Recall that on the principal subspace $W(\hat{\Lambda})$, we have the constraint $X_{(k+1)\beta}(z) = 0$, which of course implies $\Psi_{(k+1)\beta}(z) = 0$. In contrast to this mutually shared constraint, the above constraint (2.32) is a purely parafermionic phenomenon with no analog in the principal subspace. It tells us that the maximal allowed charge of the (defined below) quasi-particles generating the parafermionic space is $k-1$. This is also the maximal allowed charge of the quasi-particles generating a principal subspace at level $k-1$. So, not surprisingly, the constructed below basis for a parafermionic space at level k will be combinatorially the same as a [GeI] basis for a principal space at level $k-1$.

For the purpose of building a basis for $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}$ from the known already basis of $W(\hat{\Lambda})$, it is very natural to employ not only the parafermionic counterpart $\Psi_{r\beta}(z)$ of the bosonic current $X_{r\beta}(z)$, but also the parafermionic counterpart of a whole product of bosonic currents (with *different* variables) which differs from the product of the respective parafermionic counterparts: For given roots $\beta_r, \ldots, \beta_1 \in \Delta$ and corresponding sequence of charges $n_r, \ldots, n_1 \subset \mathbb{Z}_+$, set

$$\Psi_{n_r\beta_r,\ldots,n_1\beta_1}(z_r, \ldots, z_1) := \tag{2.33}$$

$$:= \left(\prod_{\substack{l,p=1 \\ l>p}}^{r} (z_l - z_p)^{\frac{\langle n_l\beta_l, n_p\beta_p\rangle}{k}} \right) \Psi_{n_r\beta_r}(z_r) \cdots \Psi_{n_1\beta_1}(z_1),$$

where the binomial terms are to be expanded as usual in nonnegative integral powers of the second variable. Just like in (2.19), they are inserted in order to ensure that the composition $\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot X_{n_r\beta_r}(z_r) \cdots X_{n_1\beta_1}(z_1)$ equals (up a nonzero constant) $\Psi_{n_r\beta_r,\ldots,n_1\beta_1}(z_r, \ldots z_1)$. In other words, analogously to (2.20), one can show that

$$\Psi_{n_r\beta_r,\ldots,n_1\beta_1}(z_r, \ldots z_1) = \mathrm{const} \prod_{l=1}^{r} \left(\underbrace{E^-(\frac{h_{\beta_l}}{k}, z) \otimes \cdots \otimes E^-(\frac{h_{\beta_l}}{k}, z)}_{k\ factors} \right)^{n_l} \tag{2.34}$$

$$X_{n_r\beta_r}(z_r) \cdots X_{n_1\beta_1}(z_1) \prod_{l=1}^{r} \left(\underbrace{z_l^{-\frac{h_{\beta_l}}{k}} \varepsilon_{\beta_l}^{-\frac{1}{k}} \otimes \cdots \otimes z_l^{-\frac{h_{\beta_l}}{k}} \varepsilon_{\beta_l}^{-\frac{1}{k}}}_{k\ factors} \right)^{n_l}$$

$$\prod_{l=1}^{r} \left(\underbrace{E^+(\frac{h_{\beta_l}}{k}, z) \otimes \cdots \otimes E^+(\frac{h_{\beta_l}}{k}, z)}_{k\ factors} \right)^{n_l},$$

where const $\in \mathbb{C}^\times$. We call $\Psi_{n_r\beta_r,\ldots,n_1\beta_1}(z_r, \ldots z_1)$ a *normalized product* of the parafermionic currents $\Psi_{n_r\beta_r}(z_r), \ldots, \Psi_{n_1\beta_1}(z_1)$. The last equality guarantees that a normalized product is invariant (up to a nonzero constant) under the permutation of two adjacent variables z_l, z_{l+1} and the corresponding indices $n_l\beta_l, n_{l+1}\beta_{l+1}$ as long as $\beta_l = \beta_{l+1}$ (this is not true for the usual product of parafermionic currents). Note that for any $r \in \mathbb{Z}_+$, one has

$$\Psi_{\underbrace{\beta,\ldots,\beta}_{r\ entries}}(z, \ldots, z) = \Psi_{r\beta}(z). \tag{2.35}$$

Generalizing (2.23), one defines the "Fourier coefficients" of a normalized product as follows

$$\Psi_{n_r\beta_r,\ldots,n_1\beta_1}(z_r, \ldots z_1)\Big|_{L_\mu(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} =: \tag{2.36}$$

$$=: \sum_{m_r \in \mathbb{Z} + \frac{\langle n_r\beta_r, \mu\rangle}{k}} \cdots \sum_{m_1 \in \mathbb{Z} + \frac{\langle n_1\beta_1, \mu\rangle}{k}} \psi_{n_r\beta_r,\ldots,n_1\beta_1}(m_r, \ldots, m_1) \Big|_{L_\mu(\hat\Lambda)^{\hat{\mathfrak{h}}^+}} z_r^{-m_r-n_r} \cdots z_1^{-m_1-n_1},$$

where $\beta \in \Delta$, $r \in \mathbb{Z}_+$ and $L_\mu(\hat\Lambda)^{\hat{\mathfrak{h}}^+}$ is as always the μ-weight subspace of $L(\hat\Lambda)^{\hat{\mathfrak{h}}^+}$, $\mu \in P$. We call $\psi_{n_r\beta_r,\ldots,n_1\beta_1}(m_r, \ldots, m_1)$ a *normalized ψ-monomial* or simply a *normalized monomial*, as opposed to the usual (ψ-) monomial $\psi_{n_r\beta_r}(m_r) \cdots \psi_{n_1\beta_1}(m_1)$. According to the very definition (2.33), a normalized monomial is a linear combination of usual monomials and vice versa (cf. (3.14), (3.15) below; no need to mention that our monomials are always acting on a designated space and the linear combinations in question are truncated, i.e., finite). The conformal energy of a normalized monomial is given by the following corollary of (2.24) and (2.33):

$$\Big[D - D^{\hat{\mathfrak{h}}}, \psi_{n_r\beta_r,\ldots,n_1\beta_1}(m_r, \ldots, m_1)\Big] = \tag{2.37}$$

$$= -\sum_{l=1}^{r}(m_l + \frac{n_l^2}{k} + \frac{\langle n_l\beta_l, \sum_{p<l} n_p\beta_p}{k})\psi_{n_r\beta_r,\ldots,n_1\beta_1}(m_r, \ldots, m_1).$$

As expected from the preceding discussion, the normalized monomials acting on $L(\hat\Lambda)^{\hat{\mathfrak{h}}^+}$ are indeed the parafermionic counterparts of monomials of type $x_{n_r\beta_r}(m_1) \cdots x_{n_1\beta_1}(m_1)$: One has from (2.34) and (2.36) that

$$\pi_{L(\hat\Lambda)^{\hat{\mathfrak{h}}^+}} \cdot x_{n_r\beta_r}(m_r) \cdots x_{n_1\beta_1}(m_1) \Big|_{L_\mu(\hat\Lambda)^{\hat{\mathfrak{h}}^+}} = \tag{2.38}$$

$$= \text{const}\, \psi_{n_r\beta_r,\ldots,n_1\beta_1}(m_r + \frac{\langle n_r\beta_r, \mu\rangle}{k}, \ldots, m_1 + \frac{\langle n_1\beta_1, \mu\rangle}{k}) \Big|_{L_\mu(\hat\Lambda)^{\hat{\mathfrak{h}}^+}},$$

where const $\in \mathbb{C}^\times$ (cf. (2.25)). Since the usual ψ-monomials are linear combinations of normalized ψ-monomials, one can now conclude from (2.29) and (2.30) that

$$L(\hat\Lambda)^{\hat{\mathfrak{h}}^+} = \rho(kQ) \cdot \pi_{L(\hat\Lambda)^{\hat{\mathfrak{h}}^+}} \cdot W(\hat\Lambda). \tag{2.39}$$

and

$$L(\hat\Lambda)^{\hat{\mathfrak{h}}^+}_{kQ} = \pi_{L(\hat\Lambda)^{\hat{\mathfrak{h}}^+}_{kQ}} \cdot W(\hat\Lambda) \tag{2.40}$$

(cf. (1.4) and (1.8)). This close relationship with the principal subspace will be instrumental in the subsequent arguments.

3 Quasiparticles

Recall that in the context of the principal subspace [GeI], we restricted ourselves to vertex operators associated with the simple roots because those are perfectly enough to generate the whole space when acting on a highest weight vector $v(\hat{\Lambda})$ (Lemma 3.1 [GeI]). This property is inherited in the parafermionic picture. At our convenience, we shall often be writing the ψ-monomials acting on $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}$ as parafermionic counterparts (cf. (2.25) or (2.38)):

Lemma 3.1 *One has*

$$L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+} = \mathrm{Span}_{\mathbb{C}} \Big\{ \rho(k\alpha) \cdot \pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot x_{n_r\beta_r}(m_r) \cdots \pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot x_{n_1\beta_1}(m_1) \cdot v(\hat{\Lambda}) \Big| \tag{3.1}$$
$$|x_{n_r\beta_r}(m_r) \cdots x_{n_1\beta_1}(m_1) \text{ a monomial from } U,\ \alpha \in Q \} .$$

Equivalently,

$$L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ} = \mathrm{Span}_{\mathbb{C}} \Big\{ \psi_{n_r\beta_r}(m_r + \frac{\langle n_r\beta_r, \Lambda + \sum_{p=1}^{r-1} n_p\beta_p \rangle}{k}) \cdots \psi_{n_1\beta_1}(m_1 + \frac{\langle n_1\beta_1, \Lambda \rangle}{k}) \cdot v(\hat{\Lambda}) \Big| \tag{3.2}$$
$$|x_{n_r\beta_r}(m_r) \cdots x_{n_1\beta_1}(m_1) \text{ a monomial from } U := U(\bar{\mathfrak{n}}_{\alpha_n}) \cdots U(\bar{\mathfrak{n}}_{\alpha_1}) \} .$$

Proof Follows immediately from Lemma 3.1 [GeI], the "surjectivity" (2.39) of the projection $\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}}$ and the fact that normalized monomials are linear combinations of usual monomials of the same structure (cf. (2.25) and (2.38)). □

In other words, we are entitled again to dismiss all the nonsimple roots and fix an order in the set of simple roots.

Definition 3.1 For every simple root α_i, $1 \le i \le n$, and positive integer r, we shall say that the operator $\psi_{r\alpha_i}(m)$ from (2.23) represents a *ψ-quasi-particle of color i, charge r and energy $-m - \frac{r^2}{k}$.* If confusion with x-quasi-particles can not arise, we shall skip the prefix ψ and just talk about quasi-particles; it will be clear from the context whether ψ is an operator on $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}$ or $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}$. Abusing language, we shall say that $\psi_{\alpha_i}(m)$ is from $\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot U(\bar{\mathfrak{n}}_{\alpha_i})$ and that the ψ-monomials (normalized or not) considered in Lemma 3.1 (3.1) are from $\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot U$. Be aware that the operator $\psi_{r\alpha_i}(m)$ is defined only on those μ-weight subspaces of $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}$ (resp., $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}$), for which $m \in \mathbb{Z} + \frac{\langle r\alpha_i, \mu \rangle}{k}$.

Starting from here, we shall mostly work with (ψ-) monomials from $\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}} \cdot U$.

In complete analogy with the x-monomials from U ([GeI] Section 3), we shall say that a ψ-monomial from $\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot U$

$$\psi_{n_{r_n^{(1)},n}\alpha_n}(m_{r_n^{(1)},n}) \cdots \psi_{n_{1,n}\alpha_n}(m_{1,n}) \cdots\cdots \psi_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1}) \cdots \psi_{n_{1,1}\alpha_1}(m_{1,1}) \tag{3.3}$$

and its normalized counterpart

$$\psi_{n_{r_n^{(1)},n}\alpha_n,\ldots n_{1,n}\alpha_n;\ldots\ldots;n_{r_1^{(1)},1}\alpha_1,\ldots,n_{1,1}\alpha_1}(m_{r_n^{(1)},n},\ldots,m_{1,n};\ldots\ldots;m_{r_1^{(1)},1},\ldots,m_{1,1}), \tag{3.4}$$

are of *color-charge-type*

$$(n_{r_n^{(1)},n},\ldots,n_{1,n};\ldots;n_{r_1^{(1)},1},\ldots,n_{1,1}), \tag{3.5}$$

where

$$0 < n_{r^{(1)},i} \leq \cdots \leq n_{2,i} \leq n_{1,i} \leq K,\ \sum_{p=1}^{r_i^{(1)}} n_{p,i} = r_i,\ 1 \leq i \leq n,$$

of *color-dual-charge-type*

$$(r_n^{(1)},\ldots,r_n^{(K)};\ldots;r_1^{(1)},\ldots,r_1^{(K)}), \tag{3.6}$$

where

$$r_i^{(1)} \geq r_i^{(2)} \geq \cdots \geq r_i^{(K)} \geq 0,\ \sum_{t=1}^{K} r_i^{(t)} = r_i,\ K \in \mathbb{Z}_+,\ 1 \leq i \leq n,$$

and of *color-type* $(r_n;\ldots;r_1)$. We shall also say that the corresponding generating functions

$$\Psi_{n_{r_n^{(1)},n}\alpha_n}(z_{n_{r_n^{(1)},n}})\cdots\Psi_{n_{1,1}\alpha_1}(z_{1,1}) \tag{3.7}$$

and

$$\Psi_{n_{r_n^{(1)},n}\alpha_n,\ldots,n_{1,1}\alpha_1}(z_{n_{r_n^{(1)},n}},\ldots,z_{1,1}) \tag{3.8}$$

are of the above color-charge-type, color-dual-charge-type and color-type.

We would like to remind that no ψ-quasi-particles of charge greater than $k-1$ will appear in our parafermionic space at level k: the constraint (2.31), (2.32) asserts in particular that for every color i, $1 \leq i \leq n$, one has

$$\Psi_{k\alpha_i}(z) = \mathrm{const}\,\rho(k\alpha_i),\ \mathrm{const} \in \mathbb{C}^\times \tag{3.9}$$

as an operator on $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}$, i.e.,

$$\Psi_{k\alpha_i}(z) = \mathrm{const} \in \mathbb{C}^\times \tag{3.10}$$

as an operator on the parafermionic space $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}$. Note a curious corollary of this constraint: The inverse of the identity (2.20) implies that up to an invertible operator, every x-quasi-particle of charge k and color i acts on $L(\hat{\Lambda})$ as an operator from $U(\hat{\mathfrak{h}})$, i.e.,

$$x_{k\alpha_i}(m) = \rho(k\alpha_i)\cdot h,\ h \in U(\hat{\mathfrak{h}}), \tag{3.11}$$

(recall that $U(\hat{\mathfrak{h}})$ acts through the iterated coproduct Δ^{k-1}).

Since our ambition is to exploit results from [GeI], it is really more convenient to work with the $\hat{\mathfrak{g}} \supset \hat{\mathfrak{h}}$ subspace $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}$ rather than with the parafermionic space $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}$: the

ψ-monomial $\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot x_{n_r\beta_r}(m_r) \cdots \pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot x_{n_1\beta_1}(m_1)$ and its normalized counterpart are of given color-charge-type, color-dual-charge-type and color-type if their x-counterpart $x_{n_r\beta_r}(m_r) \cdots x_{n_1\beta_1}(m_1)$ is of these types. (Strictly speaking, the ψ-monomials acting on $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}$ do not have x-counterparts because $\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}} \cdot x_{r\beta}(m)$ is not an operator on $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}$.) Note that given a ψ-monomial from $\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot U$, we do not exactly know the quasi-particle energies of its x-counterpart unless we specify the weight subspace on which the ψ-monomial acts (cf. (2.25) and (2.38)). If one discusses only the type of a monomial, this is not necessary.

In the same flow of thoughts, it is clear how to convey the linear ordering "$<$" and the partial ordering "$\prec$" from the set of x-monomials of a given color-type $(r_n; \ldots; r_1)$ (cf. [GeI] Section 3) to the set of ψ-monomials (acting on $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}$) of a given color-type: Set

$$\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot x_{n_r\beta_r}(m_r) \cdots \pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot x_{n_1\beta_1}(m_1) < \pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot x_{n'_r\beta_r}(m'_r) \cdots \pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot x_{n'_1\beta_1}(m'_1) \tag{3.12}$$

and

$$\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot x_{n_r\beta_r}(m_r) \cdots x_{n_1\beta_1}(m_1) < \pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot x_{n'_r\beta_r}(m'_r) \cdots x_{n'_1\beta_1}(m'_1) \tag{3.13}$$

if

$$x_{n_r\beta_r}(m_r) \cdots x_{n_1\beta_1}(m_1) < x_{n'_r\beta_r}(m'_r) \cdots x_{n'_1\beta_1}(m'_1).$$

Define analogously "$\prec$" for ψ-monomials of a given color-type. These definitions can obviously be rewritten, replacing the projections with the corresponding ψ-operators according to (2.25) and (2.38). Moreover, we do not have to specify the weight subspace on which the ψ-monomials act: Recall from [GeI] Section 3 that the quassi-particle energies affect the ordering only if the color-charge-types (and for that matter, the color-dual-charge-types) are the same, in which case the shift of the indices of the corresponding x-monomials will be the same for the two compared ψ-monomials. Keep in mind that "$\prec$" implies "$<$" but not vice versa.

We are now in position to explain why working with usual or with normalized ψ-monomials is essentially the same. Roughly speaking, the transformation matrix between them is "upper triangular". More precisely, from the definitions (2.33), (2.23),(2.36) and the identities (2.25), (2.38), we obtain for a given (usual) monomial from $\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot U$ that

$$\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot x_{n_r\beta_r}(m_r) \cdots \pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot x_{n_1\beta_1}(m_1) = \tag{3.14}$$

$$= \text{const}\ \pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot x_{n_r\beta_r}(m_r) \cdots x_{n_1\beta_1}(m_1) + \begin{array}{l}\text{a linear combination of normalized}\\ \text{monomials of the same color-charge-}\\ \text{-type and greater in the ordering "}\prec\text{",}\end{array}$$

where const $\in \mathbb{C}^\times$. Conversely, for any given normalized monomial from $\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot U$, we have

$$\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot x_{n_r\beta_r}(m_r) \cdots x_{n_1\beta_1}(m_1) = \tag{3.15}$$

$$= \text{const}\ \pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot x_{n_r\beta_r}(m_r) \cdots \pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot x_{n_1\beta_1}(m_1) + \begin{array}{l}\text{a linear combination of (usual)}\\ \text{monomials of the same color-}\\ \text{-charge-type and greater}\\ \text{in the ordering "}\prec\text{",}\end{array}$$

const $\in \mathbb{C}^{\times}$.

4 Quasi-particle basis for the parafermionic space

Recall that we are working with level $k \in \mathbb{Z}_+,\ k \geq 2$, and a highest weight

$$\hat{\Lambda} := k_0\hat{\Lambda}_0 + k_j\hat{\Lambda}_j = k\hat{\Lambda}_0 + \Lambda, \ \text{ where } \Lambda := k_j\Lambda_j, \tag{4.1}$$

for some $j,\ 1 \leq j \leq n$; $k_0, k_j \in \mathbb{N}$ and $k_0 + k_j = k$ (cf. Section 1). Our level k spaces are all realized in the tensor product of level one modules given by the homogeneous vertex operator construction ([GeI] Section 2). In order to treat the level one ingredients on equal footing, we introduced

$$j_t := \begin{cases} \ 0 \ \text{ for } 0 < t \leq k_0 \\ \ j \ \text{ for } k_0 < t \leq k = k_0 + k_j \end{cases} \tag{4.2}$$

and then the highest weight vector of all the modules under consideration $L(\hat{\Lambda})$, $W(\hat{\Lambda})$, $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}$ and $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}$ was defined as

$$v(\hat{\Lambda}) := v(\hat{\Lambda}_{j_k}) \otimes \cdots \otimes v(\hat{\Lambda}_{j_1}) = \tag{4.3}$$

$$= \underbrace{v(\hat{\Lambda}_j) \otimes \cdots \otimes v(\hat{\Lambda}_j)}_{k_j\ factors} \otimes \underbrace{v(\hat{\Lambda}_0) \otimes \cdots \otimes v(\hat{\Lambda}_0)}_{k_0\ factors},$$

where $v(\hat{\Lambda}_{j_t})$ is the highest weight vector of the level one module in the t^{th} tensor slot (counted from right to left).

The insightful reader has probably guessed already what is our basis-candidate for the level k parafermionic space $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}$ – the most intuitive choice is of course the $\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}}$-projection of this particular subset of the basis for $W(\hat{\Lambda})$ (from [GeI] Section 5), which contains only vectors generated by monomials with no quasi-particles of charge k. We shall rewrite for completeness the full definition of the prototype – the basis-generating set $\mathfrak{B}_{W(\hat{\Lambda})}$ from Definition 5.1 [GeI] – with the charge of the quasi-particles bounded by $k-1$ rather than k, and call it $\mathfrak{B}^{(k-1)}_{W(\hat{\Lambda})}$. We shall not comment here on the origin and naturality of this incomprehensible at first sight affluence of parameters. The frustrated readers are referred to the Introduction of [GeI] for simple particular cases and to the Introduction and Section 5 here for much easier to grasp character formulas associated with these bases. We only emphasize that the basis-generating set of monomials is a disjoint union along color-charge-types (3.5) or, equivalently, along the corresponding color-dual-charge-types (3.6), with upper bound for the charges $K := k-1$ (as opposed to $K = k$ in the principal subspace picture):

Definition 4.1 Fix a highest weight $\hat{\Lambda}$ as in (4.1). Set

$$\mathfrak{B}^{(k-1)}_{W(\hat{\Lambda})} := \bigsqcup_{\substack{0\leq n_{r_n^{(1)},n}\leq\cdots\leq n_{1,n}\leq k-1\\ \cdots\cdots\cdots\\ 0\leq n_{r_1^{(1)},1}\leq\cdots\leq n_{1,1}\leq k-1}} \left(\text{or, equivalently,} \quad \bigsqcup_{\substack{r_n^{(1)}\geq\cdots\geq r_n^{(k-1)}\geq 0\\ \cdots\cdots\cdots\\ r_1^{(1)}\geq\cdots\geq r_1^{(k-1)}\geq 0}}\right) \tag{4.4}$$

$$\left\{ x_{n_{r_n^{(1)},n}\alpha_n}(m_{r_n^{(1)},n})\cdots x_{n_{1,n}\alpha_n}(m_{1,n})\cdots\cdots x_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1})\cdots x_{n_{1,1}\alpha_1}(m_{1,1}) \quad \right|$$

$$\left| \begin{array}{l} m_{p,i}\in\mathbb{Z},\ 1\leq i\leq n,\ 1\leq p\leq r_i^{(1)};\\ m_{p,i}\leq \sum_{q=1}^{r_{i-1}^{(1)}}\min\{n_{p,i},n_{q,i-1}\}-\sum_{t=1}^{n_{p,i}}\delta_{i,j_t}-\sum_{p>p'>0}2\min\{n_{p,i},n_{p',i}\}-n_{p,i};\\ m_{p+1,i}\leq m_{p,i}-2n_{p,i}\ \text{ for }\ n_{p+1,i}=n_{p,i}\end{array} \right\},$$

where $r_0^{(1)} := 0$ and j_t was introduced in (4.2). Define

$$\mathfrak{B}_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} := \left\{ \pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}}\cdot x_{n_r\beta_r}(m_r)\cdots\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}}\cdot x_{n_1\beta_1}(m_1) \right| \tag{4.5}$$

$$\left| x_{n_r\beta_r}(m_r)\cdots x_{n_1\beta_1}(m_1)\in\mathfrak{B}^{(k-1)}_{W(\hat{\Lambda})}\right\}$$

and

$$\mathfrak{B}_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}} := \left\{ \psi_{n_r\beta_r}(m_r+\frac{\langle n_r\beta_r,\Lambda+\sum_{p=1}^{r-1}n_p\beta_p\rangle}{k})\cdots\psi_{n_1\beta_1}(m_1+\frac{\langle n_1\beta_1,\Lambda\rangle}{k}) \right| \tag{4.6}$$

$$\left| x_{n_r\beta_r}(m_r)\cdots x_{n_1\beta_1}(m_1)\in\mathfrak{B}^{(k-1)}_{W(\hat{\Lambda})}\right\}$$

(cf. (1.4), (1.6) and (1.8)).

Example 4.1 Consider $\mathfrak{g}=sl(3)$, i.e., $n=2$ and the vacuum highest weight $\hat{\Lambda}=2\hat{\Lambda}_0$ at level $k=2$. Denote for brevity the monomial

$$\pi_{L(2\hat{\Lambda}_0)^{\hat{\mathfrak{h}}^+}}\cdot x_{\alpha_2}(s)\cdots\pi_{L(2\hat{\Lambda}_0)^{\hat{\mathfrak{h}}^+}}\cdot x_{\alpha_1}(t)$$

by $(s_{\alpha_2}\ldots t_{\alpha_1})$. For the first few energy levels (the eigenvalues of the scaling operator $D-D^{\hat{\mathfrak{h}}}$ under the adjoint action), Table 1 in the Appendix lists the elements of $\mathfrak{B}_{L(2\hat{\Lambda}_0)^{\hat{\mathfrak{h}}^+}}$ of color-types $(1;2)$ and $(2;2)$. This Table is of course a copy of Table 1 [GeI], Appendix (listing the corresponding elements of $\mathfrak{B}^{(1)}_{W(2\hat{\Lambda}_0)}\cong\mathfrak{B}_{W(\hat{\Lambda}_0)}$) with the entries in the column "energy" shifted according to (2.12) and (2.14).

Example 4.2 Let again $\mathfrak{g} = sl(3)$ but consider the vacuum highest weight $\hat{\Lambda} = 3\hat{\Lambda}_0$ at level $k = 3$. Similarly to the previous example, denote the quasi-particle monomial

$$\pi_{L(3\hat{\Lambda}_0)^{\hat{\mathfrak{h}}+}} \cdot x_{s'\alpha_2}(s) \cdots \pi_{L(3\hat{\Lambda}_0)^{\hat{\mathfrak{h}}+}} \cdot x_{t'\alpha_1}(t)$$

by $(s_{s'\alpha_2} \ldots t_{t'\alpha_1})$. For the first few energy levels, Table 2 in the Appendix lists the elements of $\mathfrak{B}_{L(3\hat{\Lambda}_0)^{\hat{\mathfrak{h}}+}}$ of color-types $(1;2)$ and $(2;2)$. This Table is a copy of Table 2 [GeI], Appendix (listing the corresponding elements of $\mathfrak{B}^{(2)}_{W(3\hat{\Lambda}_0)} \cong \mathfrak{B}_{W(2\hat{\Lambda}_0)}$) with the entries in the column "energy" shifted according to (2.24) and (2.25).

We begin as usual with a proof of the spanning property of our basis-candidate.

Theorem 4.1 *Let $\hat{\Lambda}$ be a highest weight as in (4.1) and $v(\hat{\Lambda})$ be the corresponding highest weight vector (4.3). Then the set $\left\{\rho(k\alpha) \cdot b \cdot v(\hat{\Lambda}) \;\middle|\; b \in \mathfrak{B}_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}+}},\ \alpha \in Q\right\}$ spans the $\hat{\mathfrak{g}} \supset \hat{\mathfrak{h}}$ subspace $L(\hat{\Lambda})^{\hat{\mathfrak{h}}+}$. Equivalently, the set $\left\{b \cdot v(\hat{\Lambda}) \;\middle|\; b \in \mathfrak{B}_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}+}_{kQ}}\right\}$ spans the parafermionic space $L(\hat{\Lambda})^{\hat{\mathfrak{h}}+}_{kQ}$.*

Proof We shall prove the statement for $L(\hat{\Lambda})^{\hat{\mathfrak{h}}+}$ (the statement for $L(\hat{\Lambda})^{\hat{\mathfrak{h}}+}_{kQ}$ follows immediately from (2.25) and the definition (1.7)). In view of Lemma 3.1, it suffices to show that every vector $b \cdot v(\hat{\Lambda})$, b an usual ψ-monomial from $\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}+}} \cdot U$, is a linear combination of vectors from the proposed set.

Suppose that $b \notin \rho(kQ) \cdot \mathfrak{B}_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}+}}$. Due to the constraint (3.9), the (nonzero) quasi-particles of charge k in b (if any!) commute with all the other quasi-particles and can be moved ("exiled") all the way to the left. Using the "upper triangular" transformation (3.14), normalize the remaining ψ-monomial (which is by assumption $\notin \mathfrak{B}_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}+}}$) and apply Remark 5.1 [GeI] – the strong form of the spanning Theorem 5.1 [GeI] – for the obtained x-monomials (new quasi-particles of charge k might be generated in the process!). Then switch back from normalized to usual ψ-monomials using the inverse transformation (3.15) and return the "exiled" quasi-particles of charge k to their old places. The newly obtained (usual) ψ-monomials from $\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}+}} \cdot U$ have the same color-type and index-sum as b and moreover, since $b \notin \rho(kQ) \cdot \mathfrak{B}_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}+}}$, they are all greater than b in the ordering "$\prec$" (by Remark 5.1 [GeI]). Since there are only finitely many such ψ-monomials which do not annihilate $v(\hat{\Lambda})$, the statement follows by induction. □

Remark 4.1 The proof implies that the exact analog of Remark 5.1 [GeI] in the current setting is also true: Every vector $b \cdot v(\hat{\Lambda})$, b a usual ψ-monomial from $\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}+}} \cdot U$, $b \notin \rho(kQ) \cdot \mathfrak{B}_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}+}}$, is a linear combination of vectors of the form $b' \cdot v(\hat{\Lambda})$, $b' \in \rho(kQ) \cdot \mathfrak{B}_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}+}}$, $b' \succ b$, with b' and b having the same color-type and total index-sum.

We proceed with the independence. For the familiar highest weight $\hat{\Lambda} = \sum_{t=1}^{k} \hat{\Lambda}_{j_t}$ (cf.

(4.1) and (4.2)), define

$$\hat{\dot{\Lambda}} := \sum_{t=1}^{k-1} \hat{\Lambda}_{j_t} = \hat{\Lambda} - \hat{\Lambda}_{j_k}, \tag{4.7}$$

i.e., $\hat{\dot{\Lambda}} = (k-1)\hat{\Lambda}_0 + \dot{\Lambda}$, where $\dot{\Lambda} = \Lambda - \Lambda_{j_k}$ or equivalently, $\dot{\Lambda} = (k_j - 1)\Lambda_j \in P$ if $k_j > 0$ and $\dot{\Lambda} = 0$ otherwise. Note that $\hat{\dot{\Lambda}}$ is of the same type (4.1) as $\hat{\Lambda}$ (i.e., the results in [GeI] hold for $\hat{\dot{\Lambda}}$) and moreover, by the very definition (4.3), one has for the corresponding highest weight vectors $v(\hat{\Lambda}) = v(\hat{\Lambda}_{j_k}) \otimes v(\hat{\dot{\Lambda}})$. Summoning the projection $\pi_{U(\hat{\mathfrak{h}}^-)\cdot v(\hat{\Lambda}_j)}$ from [GeI] (2.12), one easily checks that

$$\Bigg(\pi_{U(\hat{\mathfrak{h}}^-)\cdot v(\hat{\Lambda}_{j_k})} \otimes \underbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{k-1\, factors}\Bigg) \cdot \mathfrak{B}_{W(\hat{\Lambda})} \cdot v(\hat{\Lambda}) = \tag{4.8}$$

$$= \Bigg(\pi_{U(\hat{\mathfrak{h}}^-)\cdot v(\hat{\Lambda}_{j_k})} \otimes \underbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{k-1\, factors}\Bigg) \cdot \mathfrak{B}^{(k-1)}_{W(\hat{\Lambda})} \cdot v(\hat{\Lambda}) = v(\hat{\Lambda}_{j_k}) \otimes \mathfrak{B}_{W(\hat{\dot{\Lambda}})} \cdot v(\hat{\dot{\Lambda}}),$$

(cf. Definition 4.1 and [GeI] Definition 5.1). This is because any nontrivial x-quasi-particle action on the leftmost tensor factor vanishes under the above projection due to the term e^{β} in the quasi-particle (cf. (2.1), (2.2)). But x-quasi-particles of charge k are zero unless they act on all the k tensor factors.

In the independence argument below, we shall employ the independence of the vectors from the set $\mathfrak{B}_{W(\hat{\dot{\Lambda}})} \cdot v(\hat{\dot{\Lambda}})$ (rather than $\mathfrak{B}_{W(\hat{\Lambda})} \cdot v(\hat{\Lambda})$!) which was proven in Theorem 5.2 [GeI].

Theorem 4.2 *Let $\hat{\Lambda}$ be again a highest weight as in (4.1) and $v(\hat{\Lambda})$ be the corresponding highest weight vector (4.3). Then the set $\left\{\rho(k\alpha) \cdot b \cdot v(\hat{\Lambda}) \,\middle|\, b \in \mathfrak{B}_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}},\ \alpha \in Q\right\}$ is indeed a basis for the $\hat{\mathfrak{g}} \supset \hat{\mathfrak{h}}$ subspace $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}$. Equivalently, the set $\left\{b \cdot v(\hat{\Lambda}) \,\middle|\, b \in \mathfrak{B}_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}}\right\}$ is a basis for the parafermionic space $L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}_{kQ}$.*

Proof It suffices to prove the independence of the vectors in the set $\rho(kQ) \cdot \mathfrak{B}_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot v(\hat{\Lambda})$. Let us first show that it would follow from the independence of the vectors in the set $v(\hat{\Lambda}_{j_k}) \otimes \mathfrak{B}_{W(\hat{\dot{\Lambda}})} \cdot v(\hat{\dot{\Lambda}})$ *over the ring* $U(\hat{\mathfrak{h}}^-)$ (recall that $U(\hat{\mathfrak{h}})$ acts through the $(k-1)$-iterate Δ^{k-1} of the standard coproduct Δ).

Suppose that there is a (nontrivial irreducible) linear relation among vectors from $\rho(kQ) \cdot \mathfrak{B}_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot v(\hat{\Lambda})$. Without loss of generality, one can assume that the relation does not contain factors from $\rho(kQ_-)$ (recall from [GeI] Preliminaries that $Q_- \subset Q$ is the semigroup generated by the negatives of the simple roots) and that at least one vector is from $\mathfrak{B}_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot v(\hat{\Lambda})$. An induction on the number of monomials involved shows that this

can be easily achieved by multiplying the relation with appropriate invertible operators from $\rho(kQ)$.

Normalize the ψ-monomials using the "upper triangular" relation (3.14): Observe that by (2.34), a vector $\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot b \cdot v(\hat{\Lambda})$, b a monomial from U, equals up to a nonzero constant $b \cdot v(\hat{\Lambda})$, plus a linear combination of vectors of the form $h' \cdot b' \cdot v(\hat{\Lambda})$, where $h' \in U(\hat{\mathfrak{h}}^-)$, b' is a monomial from U of the same color-type but of less index-sum than b, and in addition, $b' \succ b$. Implement the strong form of the spanning Theorem 5.1 [GeI] (cf. Remark 5.1 [GeI]) for the obtained x-monomials and apply the projection

$$\pi_{U(\hat{\mathfrak{h}}^-)\cdot v(\hat{\Lambda}_{j_k})} \otimes \underbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{k-1\, factors} \tag{4.9}$$

to the relation. (Note that it annihilates all the vectors containing a factor $\rho(k\alpha)$, $\alpha \in Q$.) Due to (4.8), the result is a linear relation among vectors from $v(\hat{\Lambda}_{j_k}) \otimes \mathfrak{B}_{W(\hat{\hat{\Lambda}})} \cdot v(\hat{\hat{\Lambda}})$, with coefficients in $U(\hat{\mathfrak{h}}^-)$. The relation is *nontrivial*: Among all the vectors from $\mathfrak{B}_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot v(\hat{\Lambda})$ in our initial relation (we have seen that without loss of generality, there is always at least one such vector), let $b \cdot v(\hat{\Lambda})$ be the one whose ψ-monomial b is smallest in the linear ordering "$<$". Then the projection (4.9) of the x-counterpart of b acting on $v(\hat{\Lambda})$, is nonzero (by (4.8) and Theorem 5.2 [GeI]) and is present in the final relation, because "$\prec$" implies "$<$".

In order to complete the proof, it remains to show that a linear relation among vectors from $v(\hat{\Lambda}_{j_k}) \otimes \mathfrak{B}_{W(\hat{\hat{\Lambda}})} \cdot v(\hat{\hat{\Lambda}})$, with coefficients in $U(\hat{\mathfrak{h}}^-)$ is impossible. We shall reach a contradiction with the independence of the vectors from $\mathfrak{B}_{W(\hat{\hat{\Lambda}})} \cdot v(\hat{\hat{\Lambda}})$ by restricting the relation to an appropriate homogeneous subspace where the action of the polynomial algebra $U(\hat{\mathfrak{h}}^-)$ (given by Δ^{k-1}) is "squeezed" to the leftmost tensor slot: Suppose that $m \in \mathbb{N}$ is the maximal degree of the polynomials from $U(\hat{\mathfrak{h}}^-)$ which are coefficients in our relation. Apply the projection

$$\pi_{U^m(\hat{\mathfrak{h}}^-)\cdot v(\hat{\Lambda}_{j_k})} \otimes \underbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{k-1\, factors}, \tag{4.10}$$

where $\pi_{U^m(\hat{\mathfrak{h}}^-)\cdot v(\hat{\Lambda}_j)}$ was introduced in (2.12) [GeI]. Since the vectors from the set $v(\hat{\Lambda}_{j_k}) \otimes \mathfrak{B}_{W(\hat{\hat{\Lambda}})} \cdot v(\hat{\hat{\Lambda}})$ are nonzero (Theorem 5.2 [GeI]), the number m is maximal with the property that the projection (4.10) does not annihilate all the vectors in the relation. The reason we would like to apply the projection (4.10) to our relation is quite transparent: It is easily seen from the explicit form of the iterated coproduct Δ^{k-1} that we shall thus obtain a *nontrivial* relation among vectors of the form $h \cdot v(\hat{\Lambda}_{j_k}) \otimes b \cdot v(\hat{\hat{\Lambda}})$, where $h \in U^m(\hat{\mathfrak{h}}^-)$ and $b \in \mathfrak{B}_{W(\hat{\hat{\Lambda}})}$. But this contradicts the independence of the vectors from the set $\mathfrak{B}_{W(\hat{\hat{\Lambda}})} \cdot v(\hat{\hat{\Lambda}})$ which was proven in Theorem 5.2 [GeI]. $\square$

5 Character formulas

We can finally reward ourselves and enjoy the entertaining world of characters associated with the above bases.

Consider an arbitrary monomial

$$\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot x_{n_r\beta_r}(m_r) \cdots \pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot x_{n_1\beta_1}(m_1) \tag{5.1}$$

from $\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot U$ of color-dual-charge-type

$$(r_n^{(1)}, \ldots, r_n^{(k-1)}; \ldots; r_1^{(1)}, \ldots, r_1^{(k-1)}), \tag{5.2}$$

where

$$r_i^{(1)} \geq r_i^{(2)} \geq \cdots \geq r_i^{(k-1)} \geq 0, \ \sum_{t=1}^{k-1} r_i^{(t)} = r_i, \ 1 \leq i \leq n,$$

and hence, of color-type $(r_n; \ldots; r_1)$ (cf. (3.6)). Set $p_i^{(s)}$ to be the number of quasi-particles of color i and charge s in our monomial, i.e., $p_i^{(s)} := r_i^{(s)} - r_i^{(s+1)}$, $1 \leq s \leq k-2$ and $p_i^{(k-1)} := r_i^{(k-1)}$, hence $r_i = \sum_{s=1}^{k-1} sp_i^{(s)}$, $i = 1, \ldots, n$. Then according to (2.37), (2.38) and (3.14), the $D - D^{\hat{\mathfrak{h}}}$-eigenvalue of the vector

$$\pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot x_{n_r\beta_r}(m_r) \cdots \pi_{L(\hat{\Lambda})^{\hat{\mathfrak{h}}^+}} \cdot x_{n_1\beta_1}(m_1) \cdot v(\hat{\Lambda}) \tag{5.3}$$

equals

$$\begin{aligned} -\sum_{l=1}^{r} m_l - \frac{1}{2k}\langle \sum_{i=1}^{n}(\sum_{s=1}^{k-1} sp_i^{(s)})\alpha_i, \sum_{i=1}^{n}(\sum_{s=1}^{k-1} sp_i^{(s)})\alpha_i \rangle - \frac{1}{k}\langle \sum_{i=1}^{n}(\sum_{s=1}^{k-1} sp_i^{(s)})\alpha_i, \Lambda \rangle = \\ = -\sum_{l=1}^{r} m_l - \frac{1}{2k}\langle \sum_{i=1}^{n} r_i\alpha_i, \sum_{i=1}^{n} r_i\alpha_i \rangle - \frac{1}{k}\langle \sum_{i=1}^{n} r_i\alpha_i, \Lambda \rangle = \\ = -\sum_{l=1}^{r} m_l - \frac{1}{2} \sum_{\substack{l,m=1,\ldots,n \\ s,t=1,\ldots,k-1}} A_{lm}C^{st}p_l^{(s)}p_m^{(t)} - \frac{k_j}{k}\sum_{s=1}^{k-1} sp_j^{(s)} = \\ = -\sum_{l=1}^{r} m_l - \frac{1}{k}(r_1^2 + \cdots + r_n^2 - r_1r_2 - \cdots - r_{n-1}r_n) - \frac{k_j}{k}r_j, \end{aligned} \tag{5.4}$$

where $(A_{lm})_{l,m=1}^{n}$ is the Cartan matrix of $\mathfrak{g}$ and $C^{st} := \frac{st}{k}$, $1 \leq s, t \leq k-1$. In view of Definition 4.1, (4.7) and Theorem 4.2, this formula provides the correction term needed to obtain $\operatorname{Tr} q^{D-D^{\hat{\mathfrak{h}}}}\Big|_{L(\hat{\Lambda})_{kQ}^{\hat{\mathfrak{h}}^+}}$ from $\operatorname{Tr} q^{D}\Big|_{\mathfrak{B}_{W(\hat{\Lambda})}^{(k-1)} \cdot v(\hat{\Lambda})} = \operatorname{Tr} q^{D}\Big|_{W(\hat{\Lambda})}$ (cf. also (4.8)).

Recall from [GeI] (5.27) that

$$\operatorname{Tr} q^{D}\Big|_{W(\hat{\Lambda})} = \tag{5.5}$$

$$= \sum_{\substack{p_1^{(1)},\dots,p_1^{(k-1)}\geq 0 \\ \dots\dots\dots\dots \\ p_n^{(1)},\dots,p_n^{(k-1)}\geq 0}} \frac{q^{\frac{1}{2}\sum_{l,m=1,\dots,n}^{s,t=1,\dots,k-1} A_{lm}B^{st}p_l^{(s)}p_m^{(t)}}}{\prod_{i=1}^n\prod_{s=1}^{k-1}(q)_{p_i^{(s)}}} q^{\sum_{s=k_0+1}^{k-1}(s-k_0)p_j^{(s)}}$$

where $B^{st} := \min\{s,t\}$, $1 \leq s,t \leq k-1$. But one can immediately check that

$$B^{st} - C^{st} = A_{st}^{(-1)},\ 1 \leq s,t \leq k-1, \tag{5.6}$$

where $(A_{st}^{(-1)})_{s,t=1}^{k-1}$ is the inverse of the Cartan matrix of $sl(k,\mathbb{C})$. The last three identities and Theorem 4.2 therefore imply

$$\mathrm{Tr}\, q^{D-D^{\hat{\mathfrak{h}}}}\Big|_{L(\hat{\Lambda})_{kQ}^{\hat{\mathfrak{h}}^+}} = \mathrm{Tr}\, q^{D-D^{\hat{\mathfrak{h}}}}\Big|_{L(k_0\hat{\Lambda}_0+k_j\hat{\Lambda}_j)_{kQ}^{\hat{\mathfrak{h}}^+}} = \tag{5.7}$$

$$= \sum_{\substack{p_1^{(1)},\dots,p_1^{(k-1)}\geq 0 \\ \dots\dots\dots\dots \\ p_n^{(1)},\dots,p_n^{(k-1)}\geq 0}} \frac{q^{\frac{1}{2}\sum_{l,m=1,\dots,n}^{s,t=1,\dots,k-1} A_{lm}A_{st}^{(-1)}p_l^{(s)}p_m^{(t)}}}{\prod_{i=1}^n\prod_{s=1}^{k-1}(q)_{p_i^{(s)}}} q^{\sum_{s=k_0+1}^{k-1}(s-k_0)p_j^{(s)}-\frac{k_j}{k}\sum_{s=1}^{k-1}sp_j^{(s)}}.$$

If we restrict our attention to the vacuum module ($\hat{\Lambda} = k\hat{\Lambda}_0$, that is $k_0 = k$ and $k_j = 0$), this formula is the $sl(n+1,\mathbb{C})$-case of the Kuniba-Nakanishi-Suzuki conjecture [KNS] (a dilogarithm proof of this particular case was announced in [Kir]).

Note that for a given weight $\mu \in P$, such that $\mu - \Lambda \in Q$, one obtains the $(D - D^{\hat{\mathfrak{h}}})$-character of the weight subspace $L_\mu(\Lambda)_{kQ}^{\hat{\mathfrak{h}}^+} \cong L_\mu(\Lambda)^{\hat{\mathfrak{h}}^+}$ if the additional restriction

$$\mu = \Lambda + \sum_{i=1}^n r_i\alpha_i \mod kQ = \Lambda + \sum_{i=1}^n(\sum_{s=1}^{k-1} sp_i^{(s)})\alpha_i \mod kQ \tag{5.8}$$

is imposed in the above formula (5.7). As far as the standard q-character is concerned, we know from (2.13) and (1.2) that for $\mu \in P$,

$$\mathrm{Tr}\, q^D\Big|_{L_\mu(\hat{\Lambda})} = \frac{q^{\frac{\langle\mu,\mu\rangle}{2k}-\frac{\langle\Lambda,\Lambda\rangle}{2k}}}{(q)_\infty^n}\mathrm{Tr}\, q^{D-D^{\hat{\mathfrak{h}}}}\Big|_{L_\mu(\hat{\Lambda})_{kQ}^{\hat{\mathfrak{h}}^+}}, \tag{5.9}$$

where $(q)_\infty := \prod_{l\geq 0}(1-q^l)$. On the other hand, one defines the string function $c_\mu^{\hat{\Lambda}}(q)$ as follows (cf. [KP] or [K] Section 12.7; departing from the tradition, we shall use subscript $\mu \in P$, rather than $\hat{\mu} = k\hat{\Lambda}_0 + \mu$):

$$c_\mu^{\hat{\Lambda}}(q) := q^{\frac{\langle\Lambda+\rho,\Lambda+\rho\rangle}{2(k+h^\vee)}-\frac{\langle\rho,\rho\rangle}{2h^\vee}-\frac{\langle\mu,\mu\rangle}{2k}}\ \mathrm{Tr}\, q^D\Big|_{L_\mu(\hat{\Lambda})} = \tag{5.10}$$

$$= q^{\frac{\langle \Lambda+2\rho,\Lambda\rangle}{2(k+h^\vee)} - \frac{1}{24}\frac{(\dim \mathfrak{g})k}{k+h^\vee} - \frac{\langle\mu,\mu\rangle}{2k}} \operatorname{Tr} q^D \Big|_{L_\mu(\hat\Lambda)} ,$$

(the last identity follows from the strange formula of Freudenthal-de Vries). Hence, from (5.9),

$$c_\mu^{\hat\Lambda}(q) = \frac{q^{\frac{\langle\Lambda+\rho,\Lambda+\rho\rangle}{2(k+h^\vee)} - \frac{\langle\rho,\rho\rangle}{2h^\vee} - \frac{\langle\Lambda,\Lambda\rangle}{2k}}}{(q)_\infty^n} \operatorname{Tr} q^{D-D^{\hat{\mathfrak{h}}}} \Big|_{L_\mu(\hat\Lambda)^{\hat{\mathfrak{h}}^+}_{kQ}} . \tag{5.11}$$

Remark 5.1 Observe that when the level k equals 2, formulas (5.11) and (5.9) (with the restriction (5.8)) provide combinatorial expression for *every* string function $c_\mu^{\hat\Lambda}(q)$ corresponding to a generic dominant integral weight $\hat\Lambda = \hat\Lambda_i + \hat\Lambda_j$, $0 \leq i, j \leq n$: Due to the cyclic automorphism (of order $n+1$) of the Dynkin diagram of $\hat{\mathfrak{g}}$, the generic string function equals (up to a power of q) a string function of the type $c_\mu^{\hat\Lambda_0+\hat\Lambda_j}(q)$, $0 \leq j \leq n$, considered above.

Armed with an explicit expression for the string functions, we are only a step away from writing the character of the whole standard module: Since $\rho(k\alpha)$ acts nontrivially only on the right factor of the decomposition (1.2), one concludes from (2.13), (2.18) that

$$[D, \rho(k\alpha)] \Big|_{L_\mu(\hat\Lambda)} = q^{\frac{\langle\mu+k\alpha,\mu+k\alpha\rangle}{2k} - \frac{\langle\mu,\mu\rangle}{2k}} \rho(k\alpha) \Big|_{L_\mu(\hat\Lambda)} = \tag{5.12}$$

$$= q^{\frac{k}{2}\langle\alpha+\frac{\mu}{k},\alpha+\frac{\mu}{k}\rangle - \frac{\langle\mu,\mu\rangle}{2k}} \rho(k\alpha) \Big|_{L_\mu(\hat\Lambda)} = q^{\frac{k}{2}\langle\alpha,\alpha\rangle + \langle\alpha,\mu\rangle} \rho(k\alpha) \Big|_{L_\mu(\hat\Lambda)} ,$$

where $\rho(k\alpha) := \underbrace{e^\alpha \otimes \ldots \otimes e^\alpha}_{k\ factors}$, $\alpha \in Q$. (cf. (1.6)). Therefore (1.2), (1.9), (1.10) and (5.10) imply that

$$\operatorname{ch} L(\hat\Lambda) = \operatorname{ch} L(k_0\hat\Lambda_0 + k_j\hat\Lambda_j) := \operatorname{Tr} q^D \prod_{i=1}^n y_i^{h_{\Lambda_i}} \Big|_{L(\hat\Lambda)} = \tag{5.13}$$

$$= \sum_{\mu \in \Lambda+Q/kQ} \operatorname{Tr} q^D \Big|_{L_\mu(\hat\Lambda)} q^{-\frac{\langle\mu,\mu\rangle}{2k}} \Theta_\mu(q,y) =$$

$$= q^{-\frac{\langle\Lambda+\rho,\Lambda+\rho\rangle}{2(k+h^\vee)} + \frac{\langle\rho,\rho\rangle}{2h^\vee}} \sum_{\mu \in \Lambda+Q/kQ} c_\mu^{\hat\Lambda}(q) \Theta_\mu(q,y),$$

where $\Theta_\mu(q,y)$ is the classical theta function of degree k (cf. e.g. [K] Chapters 12, 13):

$$\Theta_\mu(q,y) := q^{\frac{\langle\mu,\mu\rangle}{2k}} \sum_{\alpha\in Q} q^{\frac{k}{2}\langle\alpha,\alpha\rangle+\langle\alpha,\mu\rangle} \prod_{i=1}^n y_i^{\langle\Lambda_i,k\alpha+\mu\rangle} = \tag{5.14}$$

$$= \sum_{\gamma\in Q+\frac{\mu}{k}} q^{\frac{k}{2}\langle\gamma,\gamma\rangle} \prod_{i=1}^n y_i^{k\langle\Lambda_i,\gamma\rangle}.$$

Formula (5.13) is of course the familiar expression for the normalized character of standard module in terms of string functions and theta functions (cf. [K] (12.7.12)). Note that the explicit combinatorial formula for $c_\mu^{\hat\Lambda}$ is given by (5.11) and (5.7) with the additional restriction (5.8) imposed.

If we need only $\operatorname{ch} L(\hat\Lambda)$, we can avoid any reference to $D-D^{\hat{\mathfrak{h}}}$-characters and use directly the D-character (5.5) of the principal subspace $W(\hat{\dot\Lambda})$ (copied from [GeI] (5.27)) as well as the above theta function which incorporates as usual the contributions of the operators $\rho(k\alpha)$, $\alpha\in Q$. This is because by its very definition (1.4), (1.5), the projection $\pi_{L(\hat\Lambda)^{\hat{\mathfrak{h}}+}}$ is D-invariant. In other words, if we set $\dot\mu := \mu - \Lambda_{j_k}$, $\mu\in P$, like in (4.7) and denote by $Q_{(+)}$ the monoid (with 0) generated by positive roots, we have from (1.2), Definition 4.1, (4.8), Theorem 4.2, (5.5) and (5.12) that

$$\operatorname{ch} L(\hat\Lambda) = \operatorname{ch} L(k_0\hat\Lambda_0 + k_j\hat\Lambda_j) := \operatorname{Tr} q^D \prod_{i=1}^n y_i^{h_{\Lambda_i}}\Big|_{L(\hat\Lambda)} = \tag{5.15}$$

$$= \frac{1}{(q)_\infty^n} \sum_{\mu\in\Lambda+Q_{(+)}} \operatorname{Tr} q^D\Big|_{W_{\dot\mu}(\hat{\dot\Lambda})} q^{-\frac{\langle\mu,\mu\rangle}{2k}}\Theta_\mu(q,y) =$$

$$= \frac{1}{(q)_\infty^n} \sum_{\substack{p_1^{(1)},\ldots,p_1^{(k-1)}\ge 0\\ \cdots\cdots\cdots\\ p_n^{(1)},\ldots,p_n^{(k-1)}\ge 0}} \frac{q^{\frac12\sum_{l,m=1,\ldots,n}^{s,t=1,\ldots,k-1} A_{lm}B^{st}p_l^{(s)}p_m^{(t)}}}{\prod_{i=1}^n\prod_{s=1}^{k-1}(q)_{p_i^{(s)}}} q^{\sum_{s=k_0+1}^{k-1}(s-k_0)p_j^{(s)}}$$

$$\sum_{\alpha\in Q} q^{\frac{k}{2}\langle\alpha,\alpha\rangle+\langle\alpha,\Lambda+\sum_{i=1}^n r_i\alpha_i\rangle} \prod_{i=1}^n y_i^{\langle\Lambda_i,k\alpha+\Lambda+\sum_{i=1}^n r_i\alpha_i\rangle},$$

where as always $r_i = \sum_{s=1}^{k-1} sp_i^{(s)}$. Recall that $\hat\Lambda$ was defined in (4.1), $(A_{lm})_{l,m=1}^n$ is the Cartan matrix of $\mathfrak{g}$ and $B^{st} := \min\{s,t\}$, $1\le s,t\le k-1$.

This last character formula corresponds also to a semiinfinite monomial basis of $L(\hat\Lambda)$ in the spirit of Feigin and Stoyanovsky (the case $\hat{\mathfrak{g}} = \widehat{sl}(2,\mathbb{C})$ was described by them in the announcement [FS]). A proof of the particular case of (5.15) for the vacuum module ($\hat\Lambda = k\hat\Lambda_0$, i.e., $k_0 = k$, $k_j = 0$) was announced in [Kir].

Example 5.1 Let $\mathfrak{g} = sl(3,\mathbb{C}),\ k=2$. By (5.11) and (5.7) and (5.8) we have

$$c_0^{2\hat{\Lambda}_0}(q) = \frac{q^{-\frac{2}{15}}}{(q)_\infty^2} \sum_{\substack{p_1,p_2\geq 0\\ p_1,p_2 \text{ even}}} \frac{q^{\frac{1}{2}(p_1^2+p_2^2-p_1p_2)}}{(q)_{p_1}(q)_{p_2}}, \tag{5.16}$$

$$c_{\alpha_1+\alpha_2}^{2\hat{\Lambda}_0}(q) = \frac{q^{-\frac{2}{15}}}{(q)_\infty^2} \sum_{\substack{p_1,p_2\geq 0\\ p_1,p_2 \text{ odd}}} \frac{q^{\frac{1}{2}(p_1^2+p_2^2-p_1p_2)}}{(q)_{p_1}(q)_{p_2}}, \tag{5.17}$$

$$c_{\Lambda_1}^{\hat{\Lambda}_0+\hat{\Lambda}_1}(q) = \frac{q^{-\frac{1}{30}}}{(q)_\infty^2} \sum_{\substack{p_1,p_2\geq 0\\ p_1,p_2 \text{ even}}} \frac{q^{\frac{1}{2}(p_1^2+p_2^2-p_1p_2-p_1)}}{(q)_{p_1}(q)_{p_2}}, \tag{5.18}$$

$$c_{\Lambda_1+\alpha_2}^{\hat{\Lambda}_0+\hat{\Lambda}_1}(q) = \frac{q^{-\frac{1}{30}}}{(q)_\infty^2} \sum_{\substack{p_1,p_2\geq 0\\ p_1\ (p_2) \text{ even (odd)}}} \frac{q^{\frac{1}{2}(p_1^2+p_2^2-p_1p_2-p_1)}}{(q)_{p_1}(q)_{p_2}}, \tag{5.19}$$

Due to symmetries, the above string functions determine all the other string functions at level 2 (cf. for example [KP] Section 4.6; using the expressions of Kac and Peterson, the above formulas were verified on Maple up to $\mathcal{O}(50)$). In particular, due to the cyclic automorphism of the affine Dynkin diagram, one has $c_{\Lambda_1}^{\hat{\Lambda}_0+\hat{\Lambda}_1}(q) = c_{\Lambda_2}^{\hat{\Lambda}_0+\hat{\Lambda}_2}(q) = c_{\Lambda_1+\Lambda_2}^{\hat{\Lambda}_1+\hat{\Lambda}_2}(q)$. But notice that $\hat{\Lambda}_1+\hat{\Lambda}_2$ is not among the highest weights of type (4.1) for which our basis theorems and character formulas hold (cf. Remark 5.1). Thanks to the simplicity of the case, one can nevertheless find easily a combinatorial basis and write down the corresponding character formula:

$$\operatorname{Tr} q^{D-D^{\hat{\mathfrak{h}}}} \Big|_{L(\hat{\Lambda}_1+\hat{\Lambda}_2)_{2Q}^{\hat{\mathfrak{h}}^+}} = \tag{5.20}$$

$$= \sum_{p_1,p_2\geq 0} \frac{q^{\frac{1}{2}(p_1^2+p_2^2-p_1p_2)}}{(q)_{p_1}(q)_{p_2}} q^{p_1-\frac{1}{2}(p_1+p_2)} + \sum_{p_1,p_2\geq 0} \frac{q^{\frac{1}{2}[(p_1+1)^2+p_2^2-(p_1+1)p_2]}}{(q)_{p_1}(q)_{p_2}} q^{p_2-\frac{1}{2}[(p_1+1)+p_2]}.$$

The first term counts only monomials which do not include a quasi-particle $\pi_{L(\hat{\Lambda}_1+\hat{\Lambda}_2)_{2Q}^{\hat{\mathfrak{h}}^+}} \cdot x_{\alpha_1}(-1)$, while the second term counts only monomials which contain such a quasi-particle.

6 Appendix

color-type	*energy*	*basis*
$(1;2)$	$3/2$	$(1_{\alpha_2}-3_{\alpha_1}-1_{\alpha_1})$
	$5/2$	$(1_{\alpha_2}-4_{\alpha_1}-1_{\alpha_1}),(0_{\alpha_2}-3_{\alpha_1}-1_{\alpha_1})$
	$7/2$	$(1_{\alpha_2}-5_{\alpha_1}-1_{\alpha_1}),(1_{\alpha_2}-4_{\alpha_1}-2_{\alpha_1}),(0_{\alpha_2}-4_{\alpha_1}-1_{\alpha_1}),$ $(1_{\alpha_2}-3_{\alpha_1}-1_{\alpha_1})$
$(2;2)$	2	$(-1_{\alpha_2}1_{\alpha_2}-3_{\alpha_1}-1_{\alpha_1})$
	3	$(-2_{\alpha_2}1_{\alpha_2}-3_{\alpha_1}-1_{\alpha_1}),(-1_{\alpha_2}1_{\alpha_2}-4_{\alpha_1}-1_{\alpha_1})$
	4	$(-3_{\alpha_2}1_{\alpha_2}-3_{\alpha_1}-1_{\alpha_1}),(-2_{\alpha_2}1_{\alpha_2}-4_{\alpha_1}-1_{\alpha_1}),(-2_{\alpha_2}0_{\alpha_2}-3_{\alpha_1}-1_{\alpha_1}),$ $(-1_{\alpha_2}1_{\alpha_2}-5_{\alpha_1}-1_{\alpha_1}),(-1_{\alpha_2}1_{\alpha_2}-4_{\alpha_1}-2_{\alpha_1})$

Table 1

color-type	*energy*	*color-charge-type*	*basis*
$(1;2)$	1	$(1;2)$	$(0_{\alpha_2}-2_{2\alpha_1})$
	2	$(1;1,1)$	$(1_{\alpha_2}-3_{\alpha_1}-1_{\alpha_1})$
		$(1;2)$	$(0_{\alpha_2}-3_{2\alpha_1}),(-1_{\alpha_2}-2_{2\alpha_1})$
	3	$(1;1,1)$	$(1_{\alpha_2}-4_{\alpha_1}-1_{\alpha_1}),(0_{\alpha_2}-3_{\alpha_1}-1_{\alpha_1})$
		$(1;2)$	$(0_{\alpha_2}-4_{2\alpha_1}),(-1_{\alpha_2}-3_{2\alpha_1}),(-2_{\alpha_2}-2_{2\alpha_1})$
	4	$(1;1,1)$	$(1_{\alpha_2}-4_{\alpha_1}-2_{\alpha_1}),(1_{\alpha_2}-5_{\alpha_1}-1_{\alpha_1}),$ $(0_{\alpha_2}-4_{\alpha_1}-1_{\alpha_1}),(-1_{\alpha_2}-3_{\alpha_1}-1_{\alpha_1})$
		$(1;2)$	$(0_{\alpha_2}-5_{2\alpha_1}),(-1_{\alpha_2}-4_{2\alpha_1}),(-2_{\alpha_2}-3_{2\alpha_1}),$ $(-3_{\alpha_2}-2_{2\alpha_1})$
$(2;2)$	$2/3$	$(2;2)$	$(0_{2\alpha_2}-2_{2\alpha_1})$
	$5/3$	$(2;2)$	$(0_{2\alpha_2}-3_{2\alpha_1}),(-1_{2\alpha_2}-2_{2\alpha_1})$
	$8/3$	$(1,1;1,1)$	$(-1_{\alpha_2}1_{\alpha_2}-3_{\alpha_1}-1_{\alpha_1})$
		$(2;1,1)$	$(0_{2\alpha_2}-3_{\alpha_1}-1_{\alpha_1})$
		$(1,1;2)$	$(-2_{\alpha_2}0_{\alpha_2}-2_{2\alpha_1})$
		$(2;2)$	$(0_{2\alpha_2}-4_{2\alpha_1}),(-1_{2\alpha_2}-3_{2\alpha_1}),(-2_{2\alpha_2}-2_{2\alpha_1})$
	$11/3$	$(1,1;1,1)$	$(-1_{\alpha_2}1_{\alpha_2}-4_{\alpha_1}-1_{\alpha_1}),(-2_{\alpha_2}1_{\alpha_2}-3_{\alpha_1}-1_{\alpha_1})$
		$(2;1,1)$	$(0_{2\alpha_2}-4_{\alpha_1}-1_{\alpha_1}),(-1_{2\alpha_2}-3_{\alpha_1}-1_{\alpha_1})$
		$(1,1;2)$	$(-2_{\alpha_2}0_{\alpha_2}-3_{2\alpha_1}),(-3_{\alpha_2}0_{\alpha_2}-2_{2\alpha_1})$
		$(2;2)$	$(0_{2\alpha_2}-5_{2\alpha_1}),(-1_{2\alpha_2}-4_{2\alpha_1}),(-2_{2\alpha_2}-3_{2\alpha_1}),$ $(-3_{2\alpha_2}-2_{2\alpha_1})$

Table 2

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Dept. of Mathematics, Rutgers University, New Brunswick, NJ 08903
georgiev@math.rutgers.edu